1 Surfaces

Let $U$ be a region in $\mathbb{R}^2$ and let $H$ be an injective mapping carrying $U$ to $\mathbb{R}^3$. Let $S := H(U)$ be the range of $H$, a subset of $\mathbb{R}^3$. We will refer to $S$ as a surface in $\mathbb{R}^3$, parametrized by $H$. We will represent members of $\mathbb{R}^2$ as follows:

$$u = (u^1, u^2)$$

and members of $\mathbb{R}^3$ as follows:

$$x = (x^1, x^2, x^3)$$

Now the mapping $H$ can be expressed in the following form:

$$H(u) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$$

We will represent the total derivative of $H$ at $u$ as follows:

$$DH(u) = \begin{pmatrix} H^1_1(u) & H^1_2(u) \\ H^2_1(u) & H^2_2(u) \\ H^3_1(u) & H^3_2(u) \end{pmatrix}$$

which is to say that:

$$H^a_j(u^1, u^2) := \frac{\partial x^a}{\partial u^j}(u^1, u^2) \quad (1 \leq j \leq 2, \ 1 \leq a \leq 3)$$

We require that, for each $u$ in $U$, the column vectors:

$$H_1(u) := \begin{pmatrix} H^1_1(u) \\ H^2_1(u) \\ H^3_1(u) \end{pmatrix} \quad \text{and} \quad H_2(u) := \begin{pmatrix} H^1_2(u) \\ H^2_2(u) \\ H^3_2(u) \end{pmatrix}$$

be linearly independent, which is to say that:

$$H_1(u) \times H_2(u) \neq 0$$
2° Let $N(u)$ be the unit vector normal to the surface $S$ at the point $H(u)$:

\[
N(u) := \frac{1}{\|H_1(u) \times H_2(u)\|} (H_1(u) \times H_2(u))
\]
We define the first fundamental form $G$ for the surface $S$ as follows:

$$G(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix}$$

where:

$$G_{k\ell}(u) := H_k(u) \cdot H_\ell(u) \quad (1 \leq k \leq 2, \ 1 \leq \ell \leq 2)$$

One should note that $G(u)$ is a symmetric positive definite matrix.

We plan to describe the various metric properties of the surface $S$, such as the length of a curve in $S$, the area of a subset of $S$, and the curvature of $S$ at a point. We will show that these properties can all be expressed in terms of the first fundamental form. This fact releases us from the view that, in general, a surface must lie in $\mathbb{R}^3$. We may focus our attention upon the region $U$ in $\mathbb{R}^2$ and the first fundamental form $G$:

$$G(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix}$$

with which it has in some fashion been supplied. We may then proceed to calculate the various metric properties of $U$ in terms of $G$.

Now let $J$ be an open interval in $\mathbb{R}$ and let $\Gamma$ be a mapping carrying $J$ to $\mathbb{R}^3$ such that the range $C := \Gamma(J)$ of $\Gamma$ is a subset of the surface $S$. We require that, for each $t$ in $J$, $D\Gamma(t) \neq 0$. We shall refer to $C$ as a curve in $S$, parametrized by $\Gamma$. Of course, we may introduce the mapping $\gamma$ carrying $J$ to $U$:

$$t \longrightarrow \gamma(t) = u = (u^1(t), u^2(t))$$

such that:

$$(\Gamma^1(t), \Gamma^2(t), \Gamma^3(t)) = \Gamma(t)$$

$$= H(\gamma(t))$$

$$= (H^1(u^1(t), u^2(t)), H^2(u^1(t), u^2(t)), H^3(u^1(t), u^2(t)))$$

The mapping $\gamma$ describes the given curve $C$ in terms of the parameters $u^1$ and $u^2$. By the Chain Rule, we have:

$$D\Gamma(t) = DH(\gamma(t))D\gamma(t)$$

Hence:

$$\frac{d\Gamma}{dt}(t) = \frac{du^j}{dt}(t)H_j(\gamma(t))$$
For the latter relation, we have invoked the *summation convention*, which directs that indices which appear in a given expression both “up” and “down” shall be summation indices running through their given range (in this case, from 1 to 2). In turn:

\[ \| \frac{d\Gamma}{dt}(t) \|^2 = \frac{du_k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du_\ell}{dt}(t) \]

Now we may proceed to calculate the *length* of the segment of the curve \( C \) in \( S \) from \( \Gamma(t') \) to \( \Gamma(t'') \):

\[ (6) \quad \int_{t'}^{t''} \| D\Gamma(t) \| dt = \int_{t'}^{t''} \sqrt{\frac{du_k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du_\ell}{dt}(t)} dt \]

where \( t' \) and \( t'' \) are any numbers in \( J \) for which \( t' \leq t'' \). We are led to interpret:

\[ (7) \quad \| V \| := \sqrt{V^kG_{k\ell}(u)V^\ell} \]

as the *length* of the tangent vector:

\[ V := \begin{pmatrix} V^1 \\ V^2 \end{pmatrix} \]

to \( U \) at \( u \), and to interpret:

\[ \int_{t'}^{t''} \sqrt{\frac{du_k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du_\ell}{dt}(t)} dt \]

as the *length* of the segment of the curve \( \gamma \) in \( U \) from \( \gamma(t') \) to \( \gamma(t'') \). More generally, we interpret:

\[ (8) \quad V \circ W := V^kG_{k\ell}(u)W^\ell \]

as the *inner product* of the vectors:

\[ V = \begin{pmatrix} V^1 \\ V^2 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix} \]

in \( \mathbb{R}^2 \), tangent to \( U \) at \( u \).

5° We may also proceed to calculate the *area* of a subset \( T \) of \( S \), as follows. We first present \( T \) as \( T = H(V) \), where \( V \) is a subset of \( U \). We then equate the *area* of \( T \) with the following double integral:

\[ (9) \quad \text{area}(T) := \int_{V} \int_{V} \| H_1(u^1, u^2) \times H_2(u^1, u^2) \| du^1 du^2 \]
Since:
\[
\|H_1(u) \times H_2(u)\|^2 = G_{11}(u)G_{22}(u) - G_{21}(u)G_{12}(u) =: g(u)
\]
we interpret:
\[
\text{area}(V) := \int\int_V \sqrt{g(u^1, u^2)} \, du^1 du^2
\]
(10)
as the area of the subset \( V \) of \( U \).

1 Curvature

Let us consider a particular point \( \bar{P} \):
\[
\bar{P} = (\bar{x}^1, \bar{x}^2, \bar{x}^3) = H(\bar{u}^1, \bar{u}^2)
\]
in the surface \( S \). We plan to describe the curvature of \( S \) at \( \bar{P} \). To that end, let us consider a curve \( C \) in \( S \) containing \( \bar{P} \). The curvature of \( C \) at \( \bar{P} \) derives in part from the bending of \( C \) within \( S \) and in part from the bending of \( S \) itself. One may refer to the former as the internal bending of \( C \) and to the latter as the external bending. One may say that the internal bending is a matter of free choice but that the external bending is forced upon the curve by the structure of the surface. Among all curves \( C \) in \( S \) containing \( \bar{P} \), we may consider those for which the external bending is minimum and those for which it is maximum. By definition, the gaussian curvature of the surface \( S \) at the point \( \bar{P} \) is the product of these two extreme values.

Let \( J \) be an open interval in \( \mathbb{R} \) and let \( \Gamma \) be a mapping carrying \( J \) to \( \mathbb{R}^3 \) such that \( C := \Gamma(J) \). As usual, we require that, for each \( t \) in \( J \), \( D\Gamma(t) \neq 0 \). For convenience, let \( 0 \) be in \( J \) and let \( \Gamma(0) = \bar{P} \). In turn, let \( \gamma \) be the mapping carrying \( J \) to \( U \):
\[
t \longrightarrow \gamma(t) = u = (u^1(t), u^2(t))
\]
such that:
\[
(\Gamma^1(t), \Gamma^2(t), \Gamma^3(t)) = \Gamma(t)
= H(\gamma(t))
= (H^1(u^1(t), u^2(t)), H^2(u^1(t), u^2(t)), H^3(u^1(t), u^2(t)))
\]
Of course, \( \gamma(0) = \bar{u} = (\bar{u}^1, \bar{u}^2) \). We have:
\[
\frac{d\Gamma}{dt}(t) = \frac{du^j}{dt}(t).H_j(\gamma(t))
\]
and:
\[
\frac{d^2 \Gamma}{dt^2}(t) = \frac{d^2 u^j}{dt^2}(t).H_j(\gamma(t)) + \frac{du^k}{dt}(t)\frac{du^\ell}{dt}(t).H_{k\ell}(\gamma(t))
\]
where:
\[
H_{k\ell}(u) := \frac{\partial^2 H}{\partial u^k \partial u^\ell}(u)
\]

Now we may introduce functions $K^j_{k\ell}$ and $L_{k\ell}$ such that:
\[
H_{k\ell}(u) = K^j_{k\ell}(u).H_j(u) + L_{k\ell}(u).N(u)
\]
The foregoing relations are called Gauss’ Equations. One should note carefully that:
\[
L_{k\ell}(u) = H_{k\ell}(u) \cdot N(u)
\]

One refers to $L$:
\[
L(u) = \begin{pmatrix}
L_{11}(u) & L_{12}(u) \\
L_{21}(u) & L_{22}(u)
\end{pmatrix}
\]
as the second fundamental form for the surface $S$. One refers to $K^1$ and $K^2$:
\[
K^1(u) = \begin{pmatrix}
K^1_{11}(u) & K^1_{12}(u) \\
K^1_{21}(u) & K^1_{22}(u)
\end{pmatrix} \quad \text{and} \quad K^2(u) = \begin{pmatrix}
K^2_{11}(u) & K^2_{12}(u) \\
K^2_{21}(u) & K^2_{22}(u)
\end{pmatrix}
\]
as the connection forms for $S$. Finally, we obtain:
\[
\frac{d^2 \Gamma}{dt^2}(t) = A^j(t).H_j(\gamma(t)) + B(t).N(\gamma(t))
\]
where:
\[
A^j(t) := \frac{d^2 u^j}{dt^2}(t) + \frac{du^k}{dt}K^j_{k\ell}(\gamma(t))(t)\frac{du^\ell}{dt}(t)
\]
and:
\[
B(t) := \frac{du^k}{dt}(t)L_{k\ell}(\gamma(t))\frac{du^\ell}{dt}(t)
\]
Clearly:
\[
A^j(t).H_j(\gamma(t))
\]
is tangent to $S$ at $H(u)$. It represents the internal bending of $C$ at $H(u)$. Moreover:
\[
B(t).N(\gamma(t))
\]
is normal to $S$ at $H(u)$. It represents the external bending of $C$ at $H(u)$. 

6
At this point, we are interested in the value of $B(0)$:

$$B(0) = \frac{du^k}{dt}(0)L_{k\ell}(\bar{u})\frac{du^\ell}{dt}(0)$$

since it measures the “external bending” of $C$ at $\bar{P}$. To set the scale of computation, we require that $C$ be parametrized by arc length. The effect of this requirement is to force:

$$\frac{du^k}{dt}(t)G_{k\ell}(\gamma(t))\frac{du^\ell}{dt}(t) = 1$$

In particular:

$$\frac{du^k}{dt}(0)G_{k\ell}(\bar{u})\frac{du^\ell}{dt}(0) = 1$$

Now we wish to study the minimum and maximum values of the quantity:

$$V^kL_{k\ell}(\bar{u})V^\ell$$

where $V$ is any vector in $\mathbb{R}^2$ meeting the condition:

$$V^kG_{k\ell}(\bar{u})V^\ell = 1$$

The product of these extreme values is the gaussian curvature for $S$ at $\bar{P}$.

Here is our problem. We have two symmetric matrices:

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

and:

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

The latter is positive definite. These matrices define functions (“quadratic forms”) as follows:

$$\lambda(V) := V^kL_{k\ell}V^\ell = (V^1 \ V^2)\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

and:

$$\gamma(V) := V^kG_{k\ell}V^\ell = (V^1 \ V^2)\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

We wish to calculate the product of the minimum and the maximum values of the quantity $\lambda(V)$, subject to the condition $\gamma(V) = 1$. By “diagonalizing”
the quadratic form $L$ relative to the (positive definite) quadratic form $G$, one can show that the foregoing product equals:

$$
\frac{L_{11} L_{22} - L_{21} L_{12}}{G_{11} G_{22} - G_{21} G_{12}}
$$

Accordingly, we define the curvature of the surface $S$ at the point $\bar{P}$ to be:

$$
\kappa_S(\bar{P}) := \frac{L_{11}(\bar{u}) L_{22}(\bar{u}) - L_{21}(\bar{u}) L_{12}(\bar{u})}{G_{11}(\bar{u}) G_{22}(\bar{u}) - G_{21}(\bar{u}) G_{12}(\bar{u})}
$$

(19)

3 Geodesics

$10^\circ$ In the foregoing section, we focussed our attention upon the “external bending” of a given curve $C$ in the surface $S$, expressed by the following vector:

$$
B(t).N(\gamma(t))
$$

and we proceeded to develop a measure of “curvature” for $S$ at a given point $\bar{P}$. Now we will focus our attention upon the “internal bending” of $C$, expressed by the following vector:

$$
A^j(t).H_j(\gamma(t))
$$

By a geodesic in $S$ we mean a curve $C$ in $S$ for which the internal bending is 0. Such a curve is “as straight as possible,” given that $S$ is curved. Clearly, $C$ is a geodesic if it satisfies the following Geodesic Equations:

$$
\frac{d^2 u^j}{dt^2} + \frac{du^k}{dt}(t) K^j_{k\ell}(\gamma(t)) \frac{du^\ell}{dt}(t) = 0 \quad (1 \leq j \leq 2)
$$

(20)

To make use of these equations, we must calculate the functions:

$$
K^j_{k\ell}
$$

It will turn out that they can be expressed in terms of the first fundamental form $G$. Hence, the geodesics in $S$ are determined by $G$. We begin by defining:

$$
K_{k\ell m}(u) := H_{k\ell}(u) \bullet H_m(u)
$$

(21)

Since:

$$
G_{km}(u) = H_k(u) \bullet H_m(u)
$$
we have:
\[
\frac{\partial G_{km}}{\partial u^\ell}(u) = \frac{\partial (H_k \bullet H_m)}{\partial u^\ell}(u) \\
= H_{k\ell}(u) \bullet H_m(u) + H_k(u) \bullet H_{m\ell}(u) \\
= K_{k\ell m}(u) + K_{m\ell k}(u)
\]

By permuting the indices, we obtain:
\[
\frac{\partial G_{km}}{\partial u^\ell}(u) = K_{k\ell m}(u) + K_{m\ell k}(u)
\]
\[
\frac{\partial G_{\ell k}}{\partial u^m}(u) = K_{\ell mk}(u) + K_{km\ell}(u)
\]
\[
\frac{\partial G_{m\ell}}{\partial u^k}(u) = K_{m\ell k}(u) + K_{\ell km}(u)
\]

Since:
\[
K_{k\ell m}(u) = K_{\ell km}(u)
\]
we obtain:

(22) \[ K_{k\ell m}(u) = \frac{1}{2} \left( \frac{\partial G_{km}}{\partial u^\ell}(u) + \frac{\partial G_{\ell m}}{\partial u^k}(u) - \frac{\partial G_{k\ell}}{\partial u^m}(u) \right) \]

Now we observe that:
\[
K_{k\ell m}(u) := H_{k\ell}(u) \bullet H_m(u) \\
= K_{k\ell}^i(u) (H_i(u) \bullet H_m(u)) \\
= K_{k\ell}^i(u) G_{im}(u)
\]

Let us introduce the companion \( \hat{G} \) to \( G \), defined by inversion as follows:

(24) \[
\hat{G}(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix} = \frac{1}{g(u)} \begin{pmatrix} G_{22}(u) & -G_{12}(u) \\ -G_{21}(u) & G_{11}(u) \end{pmatrix}
\]

Clearly:

(25) \[
G_{im}(u) G^{mj}(u) = \Delta^i_j(u) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

Hence:
\[
K_{k\ell}^j(u) = K_{k\ell}^i \Delta^i_j(u) = K_{k\ell}^i(u) G_{im}(u) G^{mj}(u) = K_{k\ell m}(u) G^{mj}(u)
\]

so that:

(26) \[
K_{k\ell}^j(u) = \frac{1}{2} G^{jm}(u) \left( \frac{\partial G_{km}}{\partial u^\ell}(u) + \frac{\partial G_{\ell m}}{\partial u^k}(u) - \frac{\partial G_{k\ell}}{\partial u^m}(u) \right)
\]

These relations express the connection forms \( K^1 \) and \( K^2 \) in terms of the first fundamental form \( G \).
4 The Great Theorem of Gauss

11° Now we contend that the curvature of \( S \) at any point \( \bar{P} \) can be computed in terms of the connection forms \( K^1 \) and \( K^2 \), and the first fundamental form \( G \), hence (by the foregoing relations (26)), in terms of the first fundamental form \( G \) alone. To simplify the following computations, we will surpress reference to the variable position \( \bar{u} \) in \( U \). We begin by defining:

\[
H_{k\ell m} := \frac{\partial^3 H}{\partial u^k \partial u^\ell \partial u^m} = \frac{\partial H_{k\ell}}{\partial u^m}
\]

and:

\[
N_m := \frac{\partial N}{\partial u^m}
\]

From Gauss’ Equations – that is, from relations (12):

\[
H_{k\ell} = K^j_{k\ell} H_j + L_{k\ell} N
\]

we obtain:

\[
H_{k\ell m} = \frac{\partial K^j_{k\ell}}{\partial u^m} H_j + \frac{\partial K^j_{k\ell}}{\partial u^m} H_j m + \frac{\partial L_{k\ell}}{\partial u^m} N + L_{k\ell} N_m
\]

We must find expressions for \( N_m \). Since:

\[
N \cdot N = 1
\]

we have:

\[
N_m \cdot N = 0
\]

As a result, we may introduce coefficients \( C^\ell_m \) such that:

\[
N_m = C^\ell_m H_\ell
\]

Since:

\[
H_\ell \cdot N = 0
\]

we have:

\[
H_{k_m} \cdot N + H_k \cdot N_m = 0
\]

From relations (13):

\[
L_{km} = H_{km} \cdot N = -H_k \cdot N_m = -C^\ell_m (H_k \cdot H_\ell) = -G_{k\ell} C^\ell_m
\]

Hence:

\[
C^\ell_m = \Delta^j_k C^\ell_m = G^{jk} G_{k\ell} C^\ell_m = -G^{jk} L_{km}
\]
Finally, we obtain:

\begin{equation}
N_m = -L^j_m H_j
\end{equation}

where:

\begin{equation}
L^j_m := G^{jk} L_{km}
\end{equation}

One refers to relations (30) as Weingarten’s Equations.

12° By straightforward computation, we find that:

\[ L_{11} L_{22} - L_{21} L_{12} = (G_{11} G_{22} - G_{21} G_{12}) (L^1_1 L^2_2 - L^1_2 L^2_1) \]

Hence, we may express the gaussian curvature of \( S \) as follows:

\begin{equation}
\kappa_S = \det(L^j_m)
\end{equation}

13° Now let us return to relations (29). We have:

\begin{equation}
H_{k\ell m} = \frac{\partial K^j_{k\ell}}{\partial u^m} H_j + K^i_{k\ell} H_{im} + \frac{\partial L^j_{k\ell}}{\partial u^m} N - L^j_{k\ell} L^j_m H_j
\end{equation}

Recalling Gauss’ Equations once again, we can present the tangential and the normal components of \( H_{k\ell m} \) as follows:

\begin{equation}
H_{k\ell m} = P^j_{k\ell m} H_j + Q_{k\ell m} N
\end{equation}

where:

\begin{equation}
P^j_{k\ell m} := \frac{\partial K^j_{k\ell}}{\partial u^m} + K^i_{k\ell} K^j_{im} - L^j_{k\ell} L^j_m
\end{equation}

and:

\begin{equation}
Q_{k\ell m} := K^i_{k\ell} L_{im} + \frac{\partial L^j_{k\ell}}{\partial u^m}
\end{equation}

Since \( H_{k\ell m} = H_{k\ell m} \), we must have:

\[ P^j_{k\ell m} = P^j_{km\ell} \]

Hence:

\begin{equation}
R^j_{k\ell m} = L^j_\ell L_{km} - L^j_m L_{k\ell}
\end{equation}
where:

\begin{equation}
R^j_{k\ell m} := \left( \frac{\partial K^i_{km}}{\partial u^\ell} + K^i_{km} K^j_{i\ell} \right) - \left( \frac{\partial K^i_{k\ell}}{\partial u^m} + K^i_{k\ell} K^j_{im} \right)
\end{equation}

One refers to the functions just defined as the curvature functions for the surface \( S \). Visibly, they are defined in terms of the connection forms \( K^1 \) and \( K^2 \) for \( S \); hence, in terms of the first fundamental form \( G \) for \( S \). Finally, let us define certain companions to the curvature functions:

\begin{equation}
R^i_{k\ell m} := G_{ij} R^j_{k\ell m}
\end{equation}

By relations (36), we have:

\begin{equation}
R^i_{k\ell m} = G_{ij} (L_j^i L_{km} - L_j^i L_{k\ell}) = L_{i\ell} L_{km} - L_{im} L_{k\ell}
\end{equation}

In particular:

\begin{equation}
R^i_{1212} = L_{11} L_{22} - L_{12} L_{21}
\end{equation}

With reference to relation (19), we conclude that:

\begin{equation}
\kappa_S = \frac{R_{1212}}{g}
\end{equation}

One refers to this conclusion as “The Great Theorem” of Gauss, to the effect that one may compute the curvature of a surface \( S \) from the first fundamental form \( G \) for \( S \).

14° One can easily check that:

\begin{equation}
R_{ijk\ell} = -R_{ij\ell k}
R_{ij\ell k} = -R_{ijk\ell}
\end{equation}

Hence, the various (companion) curvature functions \( R_{ijk\ell} \) equal \(-R_{1212}, 0, \) or \( R_{1212} \). Instead of 16 different functions, we have (essentially) just one. For spaces \( S \) having dimension greater than 2, the situation is more complex.

5 Coordinate Transformations

15° The basic functions for this study are the following:

\begin{equation}
G_{k\ell}(u), \quad K^j_{k\ell}(u), \quad \text{and} \quad R^j_{k\ell m}(u)
\end{equation}

They comprise the first fundamental form, the connection forms, and the curvature form. The basic relations:

\begin{equation}
K^j_{k\ell}(u) = \frac{1}{2} G^j_{km}(u) \left( \frac{\partial G_{km}}{\partial u^\ell}(u) + \frac{\partial G_{tm}}{\partial u^k}(u) - \frac{\partial G_{k\ell}}{\partial u^m}(u) \right)
\end{equation}
\( R_{k\ell m}^l(u) = \left( \frac{\partial K^i_{k\ell}}{\partial u^m}(u) + K^i_{k\ell}(u)K^j_{lm}(u) \right) - \left( \frac{\partial K^i_{km}}{\partial u^\ell}(u) + K^i_{km}(u)K^j_{\ell m}(u) \right) \)

relate the connection forms and the curvature form to the first fundamental form. Let us consider what happens when we replace the old coordinates:

\[ u = (u^1, u^2) \]

by new coordinates:

\[ v = (v^1, v^2) \]

where:

\[ v^1 = v^1(u^1, u^2) \]
\[ v^2 = v^2(u^1, u^2) \]

and:

\[ u^1 = u^1(v^1, v^2) \]
\[ u^2 = u^2(v^1, v^2) \]

We wish to calculate:

\[ \bar{G}_{qr}(v), \quad \bar{K}_{qr}^p(v), \quad \text{and} \quad \bar{R}_{qrs}^p(v) \]

in terms of:

\[ G_{k\ell}(u), \quad K^j_{k\ell}(u), \quad \text{and} \quad R_{k\ell m}^l(u) \]

We begin by noting that:

\[ \bar{H}(v) = H(u) \]

where \( \bar{H} \) is the mapping (carrying an open subset \( V \) of \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \)) which parametrizes the surface \( S \) in terms of the new coordinates. We have:

\[ \bar{H}_q(v) = \frac{\partial u^k}{\partial v^q}(v)H_k(u) \]

Hence:

\[ (46) \quad \bar{G}_{qr}(v) = \frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)G_{k\ell}(u) \]

Since:

\[ \frac{\partial u^\ell}{\partial v^r}(v)\frac{\partial v^r}{\partial u^m}(u) = \Delta^\ell_m \]
\[ G_{km}(u)G^{mn}(u) = \Delta^n_k \]
\[ \frac{\partial v^s}{\partial u^k}(u)\frac{\partial u^k}{\partial v^q}(v) = \Delta^s_q \]
we have:
\[\left(\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)G_{kl}(u)\right)\left(\frac{\partial v^s}{\partial u^m}(u)\frac{\partial v^r}{\partial u^n}(u)G^{mn}(u)\right) = \Delta_q^s\]

Hence:
\[(47) \quad \bar{G}^{rs}(v) = \frac{\partial v^r}{\partial u^m}(u)\frac{\partial v^s}{\partial u^n}(u)G^{mn}(u)\]

By similar (but more intricate) computations, based upon relations (44), (45), (46), and (47), one can show that:
\[(48) \quad \bar{K}^p_{qr}(v) = \frac{\partial v^p}{\partial u^j}(u)\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)K^j_{kl}(u) + \frac{\partial v^p}{\partial u^m}(u)\frac{\partial^2 u^m}{\partial v^q\partial v^r}(v)\]

Moreover:
\[(49) \quad \bar{R}^p_{qrst}(v) = \frac{\partial v^p}{\partial u^j}(u)\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)\frac{\partial u^m}{\partial v^s}(v)R^j_{klm}(u)\]

and:
\[(50) \quad \bar{R}^p_{qrst}(v) = \frac{\partial u^j}{\partial v^p}(v)\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)\frac{\partial u^m}{\partial v^s}(v)R^j_{klm}(u)\]

16° As an exercise, one should show that:
\[(51) \quad \bar{R}_{1212}^1 = \kappa_S = \frac{R_{1212}}{g}\]

By relation (51), one infers that the curvature of the surface $S$ is the same, whether computed relative to the coordinates $(u^1, u^2)$ or the coordinates $(v^1, v^2)$. 