Chapter 3
Unramified Milnor–Witt K-Theories

Our aim in this section is to compute (or describe), for any integer $n > 0$, the free strongly $\mathbb{A}^1$-invariant sheaf generated by the n-th smash power of $\mathbb{G}_m$, in other words the "free strongly $\mathbb{A}^1$-invariant sheaf on n units". As we will prove in Chap. 5 that any strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is also strictly $\mathbb{A}^1$-invariant, this is also the free strictly $\mathbb{A}^1$-invariant sheaf on $(\mathbb{G}_m)^{\wedge n}$.

3.1 Milnor–Witt K-Theory of Fields

The following definition was found in collaboration with Mike Hopkins:

Definition 3.1. Let F be a commutative field. The Milnor–Witt K-theory of F is the graded associative ring $K_*^{MW}(F)$ generated by the symbols $[u]$, for each unit $u \in F^\times$, of degree $+1$, and one symbol η of degree -1 subject to the following relations:

1 (Steinberg relation) For each $a \in F^\times - \{1\} : [a].[1-a] = 0$
2 For each pair $(a,b) \in (F^\times)^2 : [ab] = [a] + [b] + \eta.[a].[b]$
3 For each $u \in F^\times : [u].\eta = \eta.[u]$
4 Set $h := \eta.[-1] + 2$. Then $\eta \ . \ h = 0$

These Milnor–Witt K-theory groups were introduced by the author in a different (and more complicated) way, until the previous presentation was found with Mike Hopkins. The advantage of this presentation was made clear in our computations of the stable $\pi_0^{\mathbb{A}^1}$ in [50, 51] as the relations all have very natural explanations in the stable $\mathbb{A}^1$-homotopical world. To perform these computations in the unstable world and also to produce unramified Milnor–Witt K-theory sheaves in a completely elementary way, over any field (any characteristic) we will need to use an "unstable" variant of that presentation in Lemma 3.4.

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Remark 3.2. The quotient ring $K_*^{MW}(F)/\eta$ is the Milnor K-theory $K_*^M(F)$ of F defined in [47]: indeed if η is killed, the symbol $[u]$ becomes additive. Observe precisely that η controls the failure of $u \mapsto [u]$ to be additive in Milnor–Witt K-theory.

With all this in mind, it is natural to introduce the Witt K-theory of F as the quotient $K_*^W(F) := K_*^{MW}(F)/h$. It was studied in [54] and will also be used in our computations below. In *loc. cit.* it was proven that the non-negative part is the quotient of the ring $Tens_{W(F)}(I(F))$ by the Steinberg relation $<< u >> . << 1-u >>$. This can be shown to still hold in characteristic 2.

Proceeding along the same line, it is easy to prove that the non-negative part $K_{\geq 0}^{MW}(F)$ is isomorphic to the quotient of the ring $Tens_{K_0^{MW}(F)}(K_1^{MW}(F))$ by the Steinberg relation $[u].[1-u]$. This is related to our old definition of $K_*^{MW}(F)$. □

We will need at some point a presentation of the group of weight n Milnor–Witt K-theory. The following one will suffice for our purpose. One may give some simpler presentation but we won't use it:

Definition 3.3. Let F be a commutative field. Let n be an integer. We let $\tilde{K}_n^{MW}(F)$ denote the abelian group generated by the symbols of the form $[\eta^m, u_1, \ldots, u_r]$ with $m \in \mathbb{N}$, $r \in \mathbb{N}$, and $n = r - m$, and with the u_i's unit in F, and subject to the following relations:

$\mathbf{1}_n$ (Steinberg relation) $[\eta^m, u_1, \ldots, u_r] = 0$ if $u_i + u_{i+1} = 1$, for some i.
$\mathbf{2}_n$ For each pair $(a,b) \in (F^\times)^2$ and each i: $[\eta^m, \ldots, u_{i-1}, ab, u_{i+1}, \ldots] = [\eta^m, \ldots, u_{i-1}, a, u_{i+1}, \ldots] + [\eta^m, \ldots, u_{i-1}, b, u_{i+1}, \ldots] + [\eta^{m+1}, \ldots, u_{i-1}, a, b, u_{i+1}, \ldots]$.
$\mathbf{4}_n$ For each i, $[\eta^{m+2}, \ldots, u_{i-1}, -1, u_{i+1}, \ldots] + 2[\eta^{m+1}, \ldots, u_{i-1}, u_{i+1}, \ldots] = 0$.

The following lemma is straightforward:

Lemma 3.4. *For any field F, any integer $n \geq 1$, the correspondence*

$$[\eta^m, u_1, \ldots, u_n] \mapsto \eta^m [u_1] \ldots [u_n]$$

induces an isomorphism

$$\tilde{K}_n^{MW}(F) \cong K_n^{MW}(F)$$

Proof. The proof consists in expressing the possible relations between elements of degree n. That is to say the element of degree n in the two-sided ideal generated by the relations of Milnor–Witt K-theory, except the number 3, which is encoded in our choices. We leave the details to the reader. □

Now we establish some elementary but useful facts. For any unit $a \in F^\times$, we set $< a >= 1 + \eta[a] \in K_0^{MW}(F)$. Observe then that $h = 1+ < -1 >$.

Lemma 3.5. *Let $(a, b) \in (F^\times)^2$ be units in F. We have the followings formulas:*

1) $[ab] = [a]+ < a > .[b] = [a]. < b > +[b]$;
2) $< ab > = < a > . < b >$; $K_0^{MW}(F)$ is central in $K_*^{MW}(F)$;
3) $< 1 >= 1$ in $K_0^{MW}(F)$ and $[1] = 0$ in $K_1^{MW}(F)$;
4) $< a >$ is a unit in $K_0^{MW}(F)$ whose inverse is $< a^{-1} >$;
5) $[\frac{a}{b}] = [a]- < \frac{a}{b} > .[b]$. In particular one has: $[a^{-1}] = - < a^{-1} > .[a]$.

Proof. (1) is obvious. One obtains the first relation of (2) by applying η to relation **2** and using relation **3**. By (1) we have for any a and b: $< a > .[b] = [b]. < a >$ thus the elements $< a >$ are central.

Multiplying relation **4** by [1] (on the left) implies that $(< 1 > -1).(< -1 > +1) = 0$ (observe that $h = 1+ < -1 >$). Using **2** this implies that $< 1 >= 1$. By (1) we have now $[1] = [1]+ < 1 > .[1] = [1] + 1.[1] = [1] + [1]$; thus $[1] = 0$. (4) follows clearly from (2) and (3). (5) is an easy consequence of (1)–(4). □

Lemma 3.6. *1) For each $n \geq 1$, the group $K_n^{MW}(F)$ is generated by the products of the form $[u_1]. \dots .[u_n]$, with the $u_i \in F^\times$.*
2) For each $n \leq 0$, the group $K_n^{MW}(F)$ is generated by the products of the form $\eta^n. < u >$, with $u \in F^\times$. In particular the product with η: $K_n^{MW}(F) \to K_{n-1}^{MW}(F)$ is always surjective if $n \leq 0$.

Proof. An obvious observation is that the group $K_n^{MW}(F)$ is generated by the products of the form $\eta^m.[u_1]. \dots .[u_\ell]$ with $m \geq 0$, $\ell \geq 0$, $\ell - m = n$ and with the u_i's units. The relation **2** can be rewritten $\eta.[a].[b] = [ab] - [a] - [b]$. This easily implies the result using the fact that $< 1 >= 1$. □

Remember that $h = 1+ < -1 >$. Set $\epsilon := - < -1 > \in K_0^{MW}(F)$. Observe then that relation **4** in Milnor–Witt K-theory can also be rewritten $\epsilon.\eta = \eta$.

Lemma 3.7. *1) For $a \in F^\times$ one has: $[a].[-a] = 0$ and $< a > + < -a >= h$;*
2) For $a \in F^\times$ one has: $[a].[a] = [a].[-1] = \epsilon[a][-1] = [-1].[a] = \epsilon[-1][a]$;
3) For $a \in F^\times$ and $b \in F^\times$ one has $[a].[b] = \epsilon.[b].[a]$;
4) For $a \in F^\times$ one has $< a^2 >= 1$.

Corollary 3.8. *The graded $K_0^{MW}(F)$-algebra $K_*^{MW}(F)$ is ϵ-graded commutative: for any element $\alpha \in K_n^{MW}(F)$ and any element $\beta \in K_m^{MW}(F)$ one has*

$$\alpha.\beta = (\epsilon)^{n.m}\beta.\alpha$$

Proof. It suffices to check this formula on the set of multiplicative generators $F^\times \amalg \{\eta\}$: for products of the form $[a].[b]$ this is (3) of the previous Lemma. For products of the form $[a].\eta$ or $\eta.\eta$, this follows from the relation **3** and relation **4** (reading $\epsilon.\eta = \eta$) in Milnor–Witt K-theory. □

Proof of Lemma 3.7. We adapt [47]. Start from the equality (for $a \neq 1$) $-a = \frac{1-a}{1-a^{-1}}$. Then $[-a] = [1-a] - < -a > .[1-a^{-1}]$. Thus

$$[a].[-a] = [a][1-a] - < -a > .[a].[1-a^{-1}] = 0 - < -a > .[a].[1-a^{-1}]$$
$$= < -a >< a > [a^{-1}][1-a^{-1}] = 0$$

by **1** and (1) of lemma 3.5. The second relation follows from this by applying η^2 and expanding.

As $[-a] = [-1] + < -1 > [a]$ we get

$$0 = [a].[-1] + < -1 > [a][a]$$

so that $[a].[a] = - < -1 > [a].[-1] = [a].[-1]$ because $0 = [1] = [-1] + < -1 > [-1]$. Using $[-a][a] = 0$ we find $[a][a] = - < -1 > [-1][a] = [-1][a]$.

Finally expanding

$$0 = [ab].[-ab] = ([a] + < a > .[b])([-a] + < -a > [b])$$

gives

$$0 = < a > ([b][-a] + < -1 > [a][b]) + < -1 > [-1][b]$$

as $[-a] = [a] + < a > [-1]$ we get

$$0 = < a > ([b][a] + < -1 > [a][b]) + [b][-1] + < -1 > [-1][b]$$

the last term is 0 by (3) so that we get the third claim.

The fourth one is obtained by expanding $[a^2] = 2[a] + \eta[a][a]$; now due to point (2) we have $[a^2] = 2[a] + \eta[-1][a] = (2 + \eta[-1])[a] = h[a]$. Applying η we thus get 0. □

Let us denote (in any characteristic) by $GW(F)$ the Grothendieck–Witt ring of isomorphism classes of non-degenerate symmetric bilinear forms [48]: this is the group completion of the commutative monoid of isomorphism classes of non-degenerate symmetric bilinear forms for the direct sum.

For $u \in F^{\times}$, we denote by $< u > \in GW(F)$ the form on the vector space of rank one F given by $F^2 \to F$, $(x, y) \mapsto uxy$. By the results of *loc. cit.*, these $< u >$ generate $GW(F)$ as a group. The following Lemma is (essentially) [48, Lemma (1.1) Chap. IV]:

Lemma 3.9. *[48] The group $GW(F)$ is generated by the elements $< u >$, $u \in F^{\times}$, and the following relations give a presentation of $GW(F)$:*

(i) $< u(v^2) > = < u >$;
(ii) $< u > + < -u > = 1 + < -1 >$;
(iii) $< u > + < v > = < u + v > + < (u + v)uv >$ *if* $(u + v) \neq 0$.

When $char(F) \neq 2$ the first two relations imply the third one and one obtains the standard presentation of the Grothendieck–Witt ring $GW(F)$, see [69]. If $char(F) = 2$ the third relation becomes $2(< u > -1) = 0$.

We observe that the subgroup (h) of $GW(F)$ generated by the hyperbolic plan $h = 1+ < -1 >$ is actually an ideal (use the relation **(ii)**). We let $W(F)$ be the quotient (both as a group or as a ring) $GW(F)/(h)$ and let $W(F) \to \mathbb{Z}/2$ be the corresponding mod 2 rank homomorphism; $W(F)$ is the Witt ring of F [48], and [69] in characteristic $\neq 2$. Observe that the following commutative square of commutative rings

$$\begin{array}{ccc} GW(F) & \to & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(F) & \to & \mathbb{Z}/2 \end{array} \tag{3.1}$$

is cartesian. The kernel of the mod 2 rank homomorphism $W(F) \to \mathbb{Z}/2$ is denoted by $I(F)$ and is called the fundamental ideal of $W(F)$.

It follows from our previous results that $u \mapsto < u > \in K_0^{MW}(F)$ satisfies all the relations defining the Grothendieck–Witt ring. Only the last one requires a comment. As the symbol $< u >$ is multiplicative in u, we may reduce to the case $u + v = 1$ by dividing by $< u + v >$ if necessary. In that case, this follows from the Steinberg relation to which one applies η^2. We thus get a ring epimorphism (surjectivity follows from Lemma 3.6)

$$\phi_0 : GW(F) \twoheadrightarrow K_0^{MW}(F)$$

For $n > 0$ the multiplication by $\eta^n : K_0^{MW}(F) \to K_{-n}^{MW}(F)$ kills h (because $h.\eta = 0$ and thus we get an epimorphism:

$$\phi_{-n} : W(F) \twoheadrightarrow K_{-n}^{MW}(F)$$

Lemma 3.10. *For each field F, each $n \geq 0$ the homomorphism ϕ_{-n} is an isomorphism.*

Proof. Following [7], let us define by $J^n(F)$ the fiber product $I^n(F) \times_{i^n(F)} K_n^M(F)$, where we use the Milnor epimorphism $s_n : K_n^M(F)/2 \twoheadrightarrow i^n(F)$, with $i^n(F) := I^n(F)/I^{(n+1)}(F)$. For $n \leq 0$, $I^n(F)$ is understood to be $W(F)$. Now altogether the $J^*(F)$ form a graded ring and we denote by $\eta \in J^{-1}(F) = W(F)$ the element $1 \in W(F)$. For any $u \in F^\times$, denote by $[u] \in J^1(F) \subset I(F) \times F^\times$ the pair $(< u > -1, u)$. Then the four relations hold in $J^*(F)$ which produces an epimorphism $K_*^{MW}(F) \twoheadrightarrow J^*(F)$. For $n > 0$ the composition of epimorphisms $W(F) \to K_{-n}^{MW}(F) \to J^{-n}(F) = W(F)$ is the identity. For $n = 0$ the composition $GW(F) \to K_0^{MW}(F) \to J^0(F) = GW(F)$ is also the identity. The Lemma is proven. $\square$

Corollary 3.11. *The canonical morphism of graded rings*

$$K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$$

induced by $[u] \mapsto \eta^{-1}(< u > -1)$ *induces an isomorphism*

$$K_*^{MW}(F)[\eta^{-1}] = W(F)[\eta, \eta^{-1}]$$

Remark 3.12. For any F let $I^*(F)$ denote the graded ring consisting of the powers of the fundamental ideal $I(F) \subset W(F)$. We let $\eta \in I^{-1}(F) = W(F)$ be the generator. Then the product with η acts as the inclusions $I^n(F) \subset I^{n-1}(F)$. We let $[u] =< u > -1 \in I(F)$ be the opposite to the Pfister form $<< u >>= 1- < u >$. Then these symbol satisfy the relations of Milnor–Witt K-theory [54] and the image of h is zero. We obtain in this way an epimorphism $K_*^W(F) \twoheadrightarrow I^*(F)$, $[u] \mapsto < u > -1 = - << u >>$. This ring $I^*(F)$ is exactly the image of the morphism $K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$ considered in the Corollary above.

We have proven that this is always an isomorphism in degree ≤ 0. In fact this remains true in degree 1, see Corollary 3.47 for a stronger version. In fact it was proven in [54] (using [1] and Voevodsky's proof of the Milnor conjectures) that

$$K_*^W(F) \twoheadrightarrow I^*(F) \tag{3.2}$$

is an isomorphism in characteristic $\neq 2$. Using Kato's proof of the analogues of those conjectures in characteristic 2 [37] we may extend this result for any field F.

From that we may also deduce (as in [54]) that the obvious epimorphism

$$K_*^W(F) \twoheadrightarrow J^*(F) \tag{3.3}$$

is always an isomorphism. □

Here is a very particular case of the last statement, but completely elementary:

Proposition 3.13. *Let F be a field for which any unit is a square. Then the epimorphism*

$$K_*^{MW}(F) \to K_*^M(F)$$

is an isomorphism in degrees ≥ 0, and the epimorphism

$$K_*^{MW}(F) \to K_*^W(F)$$

is an isomorphism in degrees < 0. In fact $I^n(F) = 0$ for $n > 0$ and $I^n(F) = W(F) = \mathbb{Z}/2$ for $n \leq 0$. In particular the epimorphisms (3.2) and (3.3) are isomorphisms.

Proof. The first observation is that $< -1 >= 1$ and thus $2\eta = 0$ (fourth relation in Milnor–Witt K-theory). Now using Lemma 3.14 below we see that for any unit $a \in F^\times$, $\eta[a^2] = 2\eta[a] = 0$, thus as any unit b is a square, we get that for any $b \in F^\times$, $\eta[b] = 0$. This proves that the second relation of Milnor–Witt K-theory gives for units (a, b) in F: $[ab] = [a] + [b] + \eta[a][b] = [a] + [b]$. The proposition now follows easily from these observations. □

Lemma 3.14. *Let $a \in F^\times$ and let $n \in \mathbb{Z}$ be an integer. Then the following formula holds in $K_1^{MW}(F)$:*

$$[a^n] = n_\epsilon[a]$$

where for $n \geq 0$, where $n_\epsilon \in K_0^{MW}(F)$ is defined as follows

$$n_\epsilon = \sum_{i=1}^{n} < (-1)^{(i-1)} >$$

(and satisfies for $n > 0$ the relation $n_\epsilon =< -1 > (n-1)_\epsilon + 1$) and where for $n \leq 0$, $n_\epsilon := - < -1 > (-n)_\epsilon$. □

Proof. The proof is quite straightforward by induction: one expands $[a^n] = [a^{n-1}] + [a] + \eta[a^{n-1}][a]$ as well as $[a^{-1}] = - < a > [a] = -([a] + \eta[a][a])$. □

3.2 Unramified Milnor–Witt *K*-Theories

In this section we will define for each $n \in \mathbb{Z}$ an explicit sheaf $\underline{\mathbf{K}}_n^{MW}$ on Sm_k called *unramified* Milnor–Witt K-theory in weight n, whose sections on any field $F \in \mathcal{F}_k$ is the group $K_n^{MW}(F)$. In the next section we will prove that for $n > 0$ this sheaf $\underline{\mathbf{K}}_n^{MW}$ is the free strongly $\mathbb{A}^1$-invariant sheaf generated by $(\mathbb{G}_m)^{\wedge n}$.

Residue homomorphisms. Recall from [47], that for any discrete valuation v on a field F, with valuation ring $\mathcal{O}_v \subset F$, and residue field $\kappa(v)$, one can define a unique homomorphism (of graded groups)

$$\partial_v : K_*^M(F) \to K_{*-1}^M(\kappa(v))$$

called "residue" homomorphism, such that

$$\partial_v(\{\pi\}\{u_2\}\ldots\{u_n\}) = \{\overline{u_2}\}\ldots\{\overline{u_n}\}$$

for any uniformizing element π and units $u_i \in \mathcal{O}_v^\times$, and where $\overline{u}$ denotes the image of $u \in \mathcal{O}_v \cap F^\times$ in $\kappa(v)$.

In the same way, given a uniformizing element π, one has:

Theorem 3.15. *There exists one and only one morphism of graded groups*

$$\partial_v^\pi : K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v))$$

which commutes to product by η *and satisfying the formulas:*

$$\partial_v^\pi([\pi][u_2]\dots[u_n]) = [\overline{u_2}]\dots[\overline{u_n}]$$

and

$$\partial_v^\pi([u_1][u_2]\dots[u_n]) = 0$$

for any units $u_1, \dots, u_n$ *of* $\mathcal{O}_v$.

Proof. Uniqueness follows from the following Lemma as well as the formulas $[a][a] = [a][-1]$, $[ab] = [a] + [b] + \eta[a][b]$ and $[a^{-1}] = - < a > [a] = -([a] + \eta[a][a])$. The existence follows from Lemma 3.16 below. □

To define the residue morphism ∂_v^π we use the method of Serre [47]. Let ξ be a variable of degree 1 which we adjoin to $K_*^{MW}(\kappa(v))$ with the relation $\xi^2 = \xi[-1]$; we denote by $K_*^{MW}(\kappa(v))[\xi]$ the graded ring so obtained.

Lemma 3.16. *Let* v *be a discrete valuation on a field* F, *with valuation ring* $\mathcal{O}_v \subset F$ *and let* π *be a uniformizing element of* v. *The map*

$$\mathbb{Z} \times \mathcal{O}_v^\times = F^\times \to K_*^{MW}(\kappa(v))[\xi]$$

$$(\pi^n.u) \mapsto \Theta_\pi(\pi^n.u) := [\overline{u}] + (n_\epsilon < \overline{u} >).\xi$$

and $\eta \mapsto \eta$ *satisfies the relations of Milnor–Witt K-theory and induce a morphism of graded rings:*

$$\Theta_\pi : K_*^{MW}(F) \to K_*^{MW}(\kappa(v))[\xi]$$

Proof. We first prove the first relation of Milnor–Witt K-theory. Let $\pi^n.u \in F^\times$ with u in $\mathcal{O}_v^\times$. We want to prove $\Theta_\pi(\pi^n.u)\Theta_\pi(1-\pi^n.u) = 0$ in $K_*^{MW}(\kappa(v))[\xi]$. If $n > 0$, then $1 - \pi^n.u$ is in $\mathcal{O}_v^\times$ and by definition $\Theta_\pi(1-\pi^n.u) = 0$. If $n = 0$, then write $1 - u = \pi^m.v$ with v a unit in $\mathcal{O}_v$. If $m > 0$ the symmetric reasoning allows to conclude. If $m = 0$, then $\Theta_\pi(u) = [\overline{u}]$ and $\Theta_\pi(1-u) = [1-\overline{u}]$ in which case the result is also clear.

It remains to consider the case $n < 0$. Then $\Theta_\pi(\pi^n.u) = [\overline{u}] + (n_\epsilon < \overline{u} >)\xi$. Moreover we write $(1-\pi^n.u)$ as $\pi^n(-u)(1-\pi^{-n}u^{-1})$ and we observe that $(-u)(1-\pi^{-n}u^{-1})$ is a unit on $\mathcal{O}_v$ so that $\Theta_\pi(1-\pi^n.u) = [-\overline{u}] + n_\epsilon < -\overline{u} > \xi$. Expanding $\Theta_\pi(\pi^n.u)\Theta_\pi(1-\pi^n.u)$ we find $[\overline{u}][-\overline{u}] + n_\epsilon < \overline{u} > \xi[-\overline{u}] + n_\epsilon < -\overline{u} > [\overline{u}][\xi] + (n_\epsilon)^2 < -1 > \xi^2$. We observe that $[\overline{u}][-\overline{u}] = 0$ and that $(n_\epsilon)^2 < -1 > \xi^2 = (n_\epsilon)^2[-1] < -1 > \xi = n_\epsilon < -1 > \xi[-1]$ because $(n_\epsilon)^2[-1] = n_\epsilon[-1]$ (this follows from Lemma 3.14 : $(n_\epsilon)^2[-1] = n_\epsilon[(-1)^n] = [(-1)^{n^2}] = [(-1)^n]$ as $n^2 - n$ is even). Thus $\Theta_\pi(\pi^n.u)\Theta_\pi(1-\pi^n.u) = n_\epsilon\{??\}\xi$

where the expression {??} is

$$< -\overline{u} > ([\overline{u}] - [-\overline{u}]) + < -1 > [-1]$$

But $[\overline{u}] - [-\overline{u}] = [\overline{u}] - [\overline{u}] - [-1] - \eta[\overline{u}][-1] = - < \overline{u} > [-1]$ thus $< -\overline{u} > ([\overline{u}] - [-\overline{u}]) = - < -1 > [-1]$, proving the result.

We now check relation 2 of Milnor–Witt K-theory. Expanding we find that the coefficient which doesn't involve ξ is 0 and the coefficient of ξ is

$$\begin{aligned} &n_\epsilon < \overline{u} > + m_\epsilon < \overline{v} > - n_\epsilon < -\overline{u} > (< \overline{v} > -1) + m_\epsilon < \overline{v} > (< u > -1) \\ &\quad + n_\epsilon m_\epsilon < \overline{uv} > (< -1 > -1) \end{aligned}$$

A careful computation (using $< \overline{u} > + < -\overline{u} > = < 1 > + < -1 > = < \overline{uv} > + < -\overline{uv} >$ yields that this term is

$$n_\epsilon + m_\epsilon - n_\epsilon m_\epsilon + < -1 > n_\epsilon m_\epsilon$$

which is shown to be $(n+m)_\epsilon$. The last two relations of the Milnor–Witt K-theory are very easy to check. □

We now proceed as in [47], we set for any $\alpha \in K_n^{MW}(F)$:

$$\Theta_\pi(\alpha) := s_v^\pi(\alpha) + \partial_v^\pi(\alpha).\xi$$

The homomorphism ∂_v^π so defined is easily checked to have the required properties. Moreover $s_v^\pi : K_*^{MW}(F) \to K_*^{MW}(\kappa(v))$ is a morphism of rings, and as such is the unique one mapping η to η and $\pi^n u$ to $[\overline{u}]$.

Proposition 3.17. *We keep the previous notations and assumptions. For any $\alpha \in K_*^{MW}(F)$:*

1) $\partial_v^\pi([-\pi].\alpha) = < -1 > s_v^\pi(\alpha)$;
2) $\partial_v^\pi([u].\alpha) = - < -1 > [\overline{u}]\partial_v^\pi(\alpha)$ for any $u \in \mathcal{O}_v^\times$.
3) $\partial_v^\pi(< u > .\alpha) = < \overline{u} > \partial_v^\pi(\alpha)$ for any $u \in \mathcal{O}_v^\times$.

Proof. We observe that, for $n \geq 1$, $K_n^{MW}(F)$ is generated as group by elements of the form $\eta^m[\pi][u_2]\dots[u_{n+m}]$ or of the form $\eta^m[u_1][u_2]\dots[u_{n+m}]$, with the u_i's units of $\mathcal{O}_v$ and with $n+m \geq 1$. Thus it suffices to check the formula on these elements, which is straightforward. □

Remark 3.18. A heuristic but useful explanation of this "trick" of Serre is the following. $Spec(F)$ is the open complement in $Spec(\mathcal{O}_v)$ of the closed point $Spec(\kappa(v))$. If one had a tubular neighborhood for that closed immersion, there should be a morphism $E(\nu_v) - \{0\} \to Spec(F)$ of the complement of the zero section of the normal bundle to $Spec(F)$; the map θ_π is the map induced in cohomology by this "hypothetical" morphism. Observe that choosing π corresponds to trivializing ν_v, in which case $E(\nu_v) - \{0\}$ becomes $(\mathbb{G}_m)_{Spec(\kappa(v))}$. Then the ring $K_*^{MW}(\kappa(v))[\xi]$ is just the ring of sections of K_*^{MW} on $(\mathbb{G}_m)_{Spec(\kappa(v))}$. The "funny" relation $\xi^2 = \xi[-1]$ which is true for any element in $K_*^{MW}(F)$, can also be explained by the fact that the reduced diagonal $(\mathbb{G}_m)_{Spec(\kappa(v))} \to (\mathbb{G}_m)^{\wedge 2}_{Spec(\kappa(v))}$ is equal to the multiplication by $[-1]$. □

Lemma 3.19. *For any field extension $E \subset F$ and for any discrete valuation on F which restricts to a discrete valuation w on E with ramification index e. Let π be a uniformizing element of v and ρ a uniformizing element of w. Write it $\rho = u\pi^e$ with $u \in \mathcal{O}_v^\times$. Then for each $\alpha \in K_*^{MW}(E)$ one has*

$$\partial_v^\pi(\alpha|_F) = e_\epsilon < \overline{u} > (\partial_w^\rho(\alpha))|_{\kappa(v)}$$

Proof. We just observe that the square (of rings)

$$\begin{array}{ccc} K_*^{MW}(F) & \overset{\Theta_\pi}{\to} & K_*^{MW}(\kappa(v))[\xi] \\ \uparrow & & \uparrow \Psi \\ K_*^{MW}(E) & \overset{\Theta_\rho}{\to} & K_*^{MW}(\kappa(w))[\xi] \end{array}$$

where Ψ is the ring homomorphism defined by $[a] \mapsto [a|_F]$ for $a \in \kappa(v)$ and $\xi \mapsto [\overline{u}] + e_\epsilon < \overline{u} > \xi$ is commutative. It is sufficient to check the commutativity in degree 1, which is not hard. □

Using the residue homomorphism and the previous Lemma one may define for any discrete valuation v on F the subgroup $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v) \subset K_n^{MW}(F)$ as the kernel of ∂_v^π. From our previous Lemma (applied to $E = F$, $e = 1$), it is clear that the kernel doesn't depend on π, only on v. We define $H_v^1(\mathcal{O}_v; \underline{\mathbf{K}}_n^{MW})$ as the quotient group $K_n^{MW}(F)/K_n^{MW}(\mathcal{O}_v)$. Once we choose a uniformizing element π we get of course a canonical isomorphism $K_n^{MW}(\kappa(v)) = H_v^1(\mathcal{O}_v; \underline{\mathbf{K}}_n^{MW})$.

Remark 3.20. One important feature of residue homomorphisms is that in the case of Milnor K-theory, these residues homomorphisms don't depend on the choice of π, only on the valuation, but in the case of Milnor–Witt K-theory, they do depend on the choice of π: for $u \in \mathcal{O}^\times$, as one has $\partial_v^\pi([u.\pi]) = \partial_v^\pi([\pi]) + \eta.[\overline{u}] = 1 + \eta.[\overline{u}]$.

This property of independence of the residue morphisms on the choice of π is a general fact (in fact equivalent) for the $\mathbb{Z}$-graded unramified sheaves M_* considered above for which the $\mathbb{Z}[F^\times/F^{\times 2}]$-structure is trivial, like Milnor K-theory. □

Remark 3.21. To make the residue homomorphisms "canonical" (see [7,8,70] for instance), one defines for a field κ and a one dimensional κ-vector space L, twisted Milnor–Witt K-theory groups: $K_*^{MW}(\kappa; L) = K_*^{MW}(\kappa) \otimes_{\mathbb{Z}[\kappa^\times]} \mathbb{Z}[L - \{0\}]$, where the group ring $\mathbb{Z}[\kappa^\times]$ acts through $u \mapsto < u >$ on $K_*^{MW}(\kappa)$ and through multiplication on $\mathbb{Z}[L-\{0\}]$. The canonical residue homomorphism is of the following form

$$\partial_v : K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v); m_v/(m_v)^2)$$

with $\partial_v([\pi].[u_2]\dots[u_n]) = [\overline{u_2}]\dots[\overline{u_n}]\otimes\overline{\pi}$, where $m_v/(m_v)^2$ is the cotangent space at v (a one dimensional $\kappa(v)$-vector space). We will make this precise in Sect. 4.1 below. □

The following result and its proof follow closely Bass–Tate [9]:

Theorem 3.22. *Let v be a discrete valuation ring on a field F. Then the subring*

$$\underline{\mathbf{K}}_*^{MW}(\mathcal{O}_v) \subset K_*^{MW}(F)$$

is as a ring generated by the elements η and $[u] \in K_1^{MW}(F)$, with $u \in \mathcal{O}_v^\times$ a unit of $\mathcal{O}_v$.

Consequently, the group $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ is generated by symbols $[u_1]\dots[u_n]$ with the u_i's in $\mathcal{O}_v^\times$ for $n\geq 1$ and by the symbols $\eta^{-n} < u >$ with the u's in $\mathcal{O}_v^\times$ for $n \leq 0$

Proof. The last statement follows from the first one as in Lemma 3.6.

We consider the quotient graded abelian group Q_* of $K_*^{MW}(F)$ by the subring A_* generated by the elements and $\eta \in K_{-1}^{MW}(F)$ and $[u] \in K_1^{MW}(F)$, with $u \in \mathcal{O}_v^\times$ a unit of $\mathcal{O}_v$. We choose a uniformizing element π. The valuation morphism induces an epimorphism $Q_* \to K_{*-1}^{MW}(\kappa(v))$. It suffices to check that this is an isomorphism. We will produce an epimorphism $K_{*-1}^{MW}(\kappa(v)) \to Q_*$ and show that the composition $K_{*-1}^{MW}(\kappa(v)) \to Q_* \to K_{*-1}^{MW}(\kappa(v))$ is the identity.

We construct a $K_*^{MW}(\kappa(v))$-module structure on $Q_*(F)$. Denote by $\mathcal{E}_*$ the graded ring of endomorphisms of the graded abelian group $Q_*(F)$. First the element η still acts on Q_* and yields an element $\eta\in\mathcal{E}_{-1}$. Let $a\in\kappa(v)^\times$ be a unit in $\kappa(v)$. Choose a lifting $\tilde{\alpha}\in\mathcal{O}_v^\times$. Then multiplication by $\tilde{\alpha}$ induces a morphism of degree $+1$, $Q_*\to Q_{*+1}$. We first claim that it doesn't depend on the choice of $\tilde{\alpha}$. Let $\tilde{\alpha}'=\beta\tilde{\alpha}$ be another lifting so that $u\in\mathcal{O}_v^\times$ is congruent to 1 mod π. Expanding $[\tilde{\alpha}'] = [\tilde{\alpha}]+[\beta]+\eta[\tilde{\alpha}][\beta]$ we see that it is sufficient to check that for any $a\in F^\times$, the product $[\beta][a]$ lies in the subring A_*. Write $a=\pi^n.u$ with $u\in\mathcal{O}_v^\times$. Then expanding $[\pi^n.u]$ we end up to checking the property for the product $[\beta][\pi^n]$, and using Lemma 3.14 we may even assume $n=1$. Write $\beta = 1-\pi^n.v$, with $n>0$ and $v\in\mathcal{O}_v^\times$.

Thus we have to prove that the products of the above form $[1-\pi^n.v][\pi]$ are in A_*. For $n=1$, the Steinberg relation yields $[1-\pi.v][\pi.v]=0$. Expanding $[\pi.v]=[\pi](1+\eta[v])+[v]$, implies $[1-\pi.v][\pi](1+\eta[v])$ is in A_*. But by Lemma 3.7, $1+\eta[v] =< v >$ is a unit of A_*, with inverse itself. Thus $[1-\pi.v][\pi]\in A_*$. Now if $n\geq 2$, $1-\pi^n.v = (1-\pi)+\pi(1-\pi^{n-1}v) = (1-\pi)(1+\pi(\frac{1-\pi^{n-1}}{1-\pi})) = (1-\pi)(1-\pi w)$, with $w\in\mathcal{O}_v^\times$. Expending, we get

$[1-\pi^n.v][\pi] = [1-\pi][\pi] + [1-\pi w][\pi] + \eta[1-\pi][1-\pi w][\pi] = [1-\pi w][\pi]$. Thus the result holds in general.

We thus define this way elements $[u] \in \mathcal{E}_1$. We now claim these elements (together with η) satisfy the four relations in Milnor–Witt K-theory: this is very easy to check, by the very definitions. Thus we get this way a $K_*^{MW}(\kappa(v))$-module structure on Q_*. Pick up the element $[\pi] \in Q_1 = K_1^{MW}(F)/A_1$. Its image through ∂_v^π is the generator of $K_*^{MW}(\kappa(v))$ and the homomorphism $K_{*-1}^{MW}(\kappa(v)) \to Q_*, \alpha \mapsto \alpha.[\pi]$ provides a section of $\partial_v^\pi : Q_* \to K_{*-1}^{MW}(\kappa(v))$. This is clear from our definitions.

It suffices now to check that $K_{*-1}^{MW}(\kappa(v)) \to Q_*$ is onto. Using the fact that any element of F can be written $\pi^n u$ for some unit $u \in \mathcal{O}_v^\times$, we see that $K_*^{MW}(F)$ is generated as a group by elements of the form $\eta^m[\pi][u_2]\dots[u_n]$ or $\eta^m[u_1]\dots[u_n]$, with the u_i's in $\mathcal{O}_v^\times$. But the latter are in A_* and the former are, modulo A_*, in the image of $K_{*-1}^{MW}(\kappa(v)) \to Q_*$. □

Remark 3.23. In fact one may also prove as in *loc. cit.* the fact that the morphism Θ_π defined in the Lemma 3.16 is onto and its kernel is the ideal generated by η and the elements $[u] \in K_1^{MW}(F)$ with $u \in \mathcal{O}_v^\times$ a unit of $\mathcal{O}_v$ congruent to 1 modulo π. We will not give the details here, we do not use these results. □

Theorem 3.24. *For any field F the following diagram is a (split) short exact sequence of $K_*^{MW}(F)$-modules:*

$$0 \to K_n^{MW}(F) \to K_n^{MW}(F(T)) \xrightarrow{\Sigma\partial_{(P)}^P} \oplus_P K_{n-1}^{MW}(F[T]/P) \to 0$$

(where P runs over the set of monic irreducible polynomials of $F[T]$).

Proof. It it is again very much inspired from [47]. We first observe that the morphism $K_*^{MW}(F) \to K_*^{MW}(F(T))$ is a split monomorphism; from our previous computations we see that $K_*^{MW}(F(T)) \xrightarrow{\partial_{(T)}^T([T]\cup -)} K_*^{MW}(F)$ provides a retraction.

Now we define a filtration on $K_*^{MW}(F(T))$ by sub-rings L_d's

$$L_0 = K_*^{MW}(F) \subset L_1 \subset \cdots \subset L_d \subset \cdots \subset K_*^{MW}(F(T))$$

such that L_d is exactly the sub-ring generated by $\eta \in K_{-1}^{MW}(F(T))$ and all the elements $[P] \in K_1^{MW}(F(T))$ with $P \in F[T] - \{0\}$ of degree less or equal to d. Thus L_0 is indeed $K_*^{MW}(F) \subset K_*^{MW}(F(T))$. Observe that $\bigcup_d L_d = K_*^{MW}(F(T))$. Observe that each L_d is actually a sub $K_*^{MW}(F)$-algebra.

Also observe that using the relation $[a.b] = [a]+[b]+\eta[a][b]$ that if $[a] \in L_d$ and $[b] \in L_d$ then so are $[ab]$ and $[\frac{a}{b}]$. As a consequence, we see that for $n \geq 1$, $L_d(K_n^{MW}(F(T)))$ is the sub-group generated by symbols $[a_1]\dots[a_n]$ such that each a_i itself is a fraction which involves only polynomials of degree

$\leq d$. In degree ≤ 0, we see in the same way that $L_d(K_n^{MW}(F(T)))$ is the subgroup generated by symbols $< a > \eta^n$ with a a fraction which involves only polynomials of degree $\leq d$.

It is also clear that for $n \geq 1$, $L_d(K_n^{MW}(F(T)))$ is generated as a group by elements of the form $\eta^m[a_1]\dots[a_{n+m}]$ with the a_i of degree $\leq d$. □

Lemma 3.25. *1) For $n \geq 1$, $L_d(K_n^{MW}(F(T)))$ is generated by the elements of $L_{(d-1)}(K_n^{MW}(F(T)))$ and elements of the form $\eta^m[a_1]\dots[a_{n+m}]$ with a_1 of degree d and the a_i's, $i \geq 2$ of degree $\leq (d-1)$.*

2) Let $P \in F[T]$ be a monic polynomial of degree $d > 0$. Let $G_1, \dots, G_i$ be polynomials of degrees $\leq (d-1)$. Finally let G be the rest of the Euclidean division of $\Pi_{j\in\{1,\dots,i\}}G_j$ by P, so that G has degree $\leq (d-1)$. Then one has in the quotient group $K_2^{MW}(F(T))/L_{d-1}$ the equality

$$[P][G_1 \dots G_i] = [P][G]$$

Proof. 1) We proceed as in Milnor's paper. Let f_1 and f_2 be polynomials of degree d. We may write $f_2 = -af_1 + g$, with $a \in F^\times$ a unit and g of degree $\leq (d-1)$. If $g = 0$, the we have $[f_1][f_2] = [f_1][a(-f_1)] = [f_1][a]$ (using the relation $[f_1, -f_1] = 0$). If $g \neq 0$ then as in *loc. cit.* we get $1 = \frac{af_1}{g} + \frac{f_2}{g}$ and the Steinberg relation yields $[\frac{af_1}{g}][\frac{f_2}{g}] = 0$. Expanding with η we get: $([f_1] - [\frac{g}{a}] - \eta[\frac{g}{a}][\frac{af_1}{g}])[\frac{f_2}{g}] = 0$, which readily implies (still in $K_2^{MW}(F(T))$):

$$([f_1] - [\frac{g}{a}])[\frac{f_2}{g}] = 0$$

But expanding the right factor now yields

$$([f_1] - [\frac{g}{a}])([f_2] - [g] - \eta[g][\frac{f_2}{g}]) = 0$$

which implies (using again the previous vanishing):

$$([f_1] - [\frac{g}{a}])([f_2] - [g]) = 0$$

We see that $[f_1][f_2]$ can be expressed as a sum of symbols in which at most one of the factor as degree d, the other being of smaller degree. An easy induction proves (1).

2) We first establish the case $i = 2$. We start with the Euclidean division $G_1G_2 = PQ + G$. We get from this the equality $1 = \frac{G}{G_1.G_2} + \frac{PQ}{G_1.G_2}$ which gives $[\frac{PQ}{G_1.G_2}][\frac{G}{G_1.G_2}] = 0$. We expand the left term as $[\frac{PQ}{G_1.G_2}] = < \frac{Q}{G_1.G_2} > [P] + [\frac{Q}{G_1.G_2}]$. We thus obtain $[P][\frac{G}{G_1.G_2}] = - < \frac{Q}{G_1.G_2} > [\frac{Q}{G_1.G_2}][\frac{G}{G_1.G_2}]$ but the right hand side is in $L_{(d-1)}$ (observe Q has degree

$\leq (d-1)$) thus $[P][\frac{G}{G_1.G_2}] \in L_{(d-1)} \subset K_2^{MW}(F(T))$. Now $[\frac{G}{G_1G_2}] = [G] - [G_1G_2] - \eta[G_1G_2][\frac{G}{G_1G_2}]$. Thus $[P][\frac{G}{G_1.G_2}] = [P][G] - [P][G_1G_2] + <-1> \eta[G_1G_2][P][\frac{G}{G_1G_2}]$. This shows that modulo $L_{(d-1)}$, $[P][G] - [P][G_1G_2]$ is zero, as required.

For the case $i \geq 3$ we proceed by induction. Let $\Pi_{j\in\{2,\ldots,i\}}G_j = P.Q + G'$ be the Euclidean division of $\Pi_{j\in\{2,\ldots,i\}}G_j$ by P with G' of degree $\leq (d-1)$. Then the rest G of the Euclidean division by P of $G_1 \ldots G_i$ is the same as the rest of the Euclidean division of G_1G' by P. Now $[P][G_1 \ldots G_i] = [P][G_1] + [P][G_2 \ldots G_i] + \eta[P][G_2 \ldots G_i][G_1]$. By the inductive assumption this is equal, in $K_2^{MW}(F(T))/L_{d-1}$, to $[P][G_1] + [P][G'] + \eta[P][G'][G_1] = [P][G'G_1]$. By the case 2 previously proven we thus get in $K_2^{MW}(F(T))/L_{d-1}$,

$$[P][G_1 \ldots G_i] = [P][G_1G'] = [P][G]$$

which proves our claim. □

Now we continue the proof of Theorem 3.24 following Milnor's proof of [47, Theorem 2.3]. Let $d \geq 1$ be an integer and let $P \in F[T]$ be a monic irreducible polynomial of degree d. We denote by $\mathcal{K}_P \subset L_d/L_{(d-1)}$ the sub-graded group generated by elements of the form $\eta^m[P][G_1]\ldots[G_n]$ with the G_i of degree $(d-1)$. For any polynomial G of degree $\leq (d-1)$, the multiplication by $\epsilon[G]$ induces a morphism:

$$\epsilon[G].: \mathcal{K}_P \to \mathcal{K}_P$$

$$\eta^m[P][G_1]\ldots[G_n] \mapsto \epsilon[G]\eta^m[P][G_1]\ldots[G_n] = \eta^m[P][G][G_1]\ldots[G_n]$$

of degree $+1$. Let $\mathcal{E}_P$ be the graded associative ring of graded endomorphisms of $\mathcal{K}_P$. We claim that the map $(F[T]/P)^\times \to (\mathcal{E}_P)_1, \overline{(G)} \mapsto \epsilon[G]$. (where G has degree $\leq (d-1)$) and the element $\eta \in (\mathcal{E}_P)_{-1}$ (corresponding to the multiplication by η) satisfy the four relations of the Milnor–Witt K-theory. Let us check the Steinberg relation. Let $G \in F[T]$ be of degree $\leq (d-1)$. Then so is $1-G$ and the relation $(\epsilon[G].) \circ (\epsilon[1-G].) = 0 \in \mathcal{E}_P$ is clear. Let us check relation **2**. We let H_1 and H_2 be polynomials of degree $\leq (d-1)$. Let G be the rest of division of H_1H_2 by P. By definition $\epsilon[\overline{(H_1)}\overline{(H_2)}]$. is $\epsilon[\overline{(G)}]$.. But by the part (2) of the Lemma we have (in $\mathcal{K}_P \subset K_m^{MW}(F(T))/L_{(d-1)}$):

$$\begin{aligned}\epsilon[\overline{(G)}].(\eta^m[P][G_1]\ldots[G_n]) &= \eta^m[P][G][G_1]\ldots[G_n] \\ &= \eta^m[P][H_1H_2][G_1]\ldots[G_n]\end{aligned}$$

which easily implies the claim. The last two relations are easy to check.

We thus obtain a morphism of graded ring $K_*^{MW}(F[T]/P) \to \mathcal{E}_P$. By letting $K_*^{MW}(F[T]/P)$ act on $[P] \in L_d/L_{(d-1)} \subset K_1^{MW}(F(T))/L_{(d-1)}$ we obtain a graded homomorphism

$$K_*^{MW}(F[T]/P) \to \mathcal{K}_P \subset L_d/L_{(d-1)}$$

which is an epimorphism. By the first part of the Lemma, we see that the induced homomorphism

$$\oplus_P K_*^{MW}(F[T]/P) \to L_d/L_{(d-1)} \tag{3.4}$$

is an epimorphism. Now using our definitions, one checks as in [47] that for P of degree d, the residue morphism ∂^P vanishes on $L_{(d-1)}$ and that moreover the composition

$$\oplus_P K_*^{MW}(F[T]/P) \twoheadrightarrow L_d(K_n^{MW}(F(T)))/L_{(d-1)}(K_n^{MW}(F(T)))$$
$$\xrightarrow{\Sigma_P \partial^P} \oplus_P K_*^{MW}(F[T]/P)$$

is the identity. As in *loc. cit.* this implies the theorem, with the observation that the quotients L_d/L_{d-1} are $K_*^{MW}(F)$-modules and the residues maps are morphisms of $K_*^{MW}(F)$-modules. □

Remark 3.26. We observe that the previous theorem in negative degrees is exactly [53, Theorem 5.3].

Now we come back to our fixed base field k and work in the category $\mathcal{F}_k$. We will make constant use of the results of Sect. 2.3. We endow the functor $F \mapsto K_*^{MW}(F)$, $\mathcal{F}_k \to \mathcal{A}b_*$ with Data **(D4) (i)**, **(D4) (ii)** and **(D4) (iii)**. The datum **(D4) (i)** comes from the $K_0^{MW}(F) = GW(F)$-module structure on each $K_n^{MW}(F)$ and the datum **(D4) (ii)** comes from the product $F^\times \times K_n^{MW}(F) \to K_{(n+1)}^{MW}(F)$. The residue homomorphisms ∂_v^π gives the Data **(D4) (iii)**. We observe of course that these Data are extended from the prime field of k.

Axioms **(B0)**, **(B1)** and **(B2)** are clear from our previous results. The Axiom **(B3)** follows at once from Lemma 3.19.

Axiom **(HA) (ii)** is clear, Theorem 3.24 establishes Axiom **(HA) (i)**.

For any discrete valuation v on $F \in \mathcal{F}_k$, and any uniformizing element π, define morphisms of the form $\partial_z^y : K_n^{MW}(\kappa(y)) \to K_{n-1}^{MW}(\kappa(z))$ for any $y \in (\mathbb{A}_F^1)^{(1)}$ and $z \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}$ fitting in the following diagram:

$$\begin{array}{ccccccc} 0 \to & K_*^{MW}(F) & \to & K_*^{MW}(F(T)) & \to \oplus_{y\in(\mathbb{A}_F^1)^{(1)}} K_{*-1}^{MW}(\kappa(y)) \to 0 \\ & \downarrow \partial_v^\pi & & \downarrow \partial_{v[T]}^\pi & \downarrow \Sigma_{y,z}\partial_z^{\pi,y} \\ 0 \to & K_{*-1}^{MW}(\kappa(v)) & \to & K_{n-1}^{MW}(\kappa(v)(T)) & \to \oplus_{z\in\mathbb{A}_{\kappa(v)}^1} K_{*-2}^{MW}(\kappa(v)) \to 0 \end{array} \tag{3.5}$$

The following Theorem establishes Axiom **(B4)**.

Theorem 3.27. *Let v be a discrete valuation on $F \in \mathcal{F}_k$, let π be a uniformizing element. Let $P \in \mathcal{O}_v[T]$ be an irreducible primitive polynomial, and $Q \in \kappa(v)[T]$ be an irreducible monic polynomial.*

(i) If the closed point $Q \in \mathbb{A}^1_{\kappa(v)} \subset \mathbb{A}^1_{\mathcal{O}_v}$ is not in the divisor D_P then the morphism $\partial_Q^{\pi,P}$ is zero.
(ii) If Q is in $D_P \subset \mathbb{A}^1_{\mathcal{O}_v}$ and if the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with π as uniformizing element then

$$\partial_Q^{\pi,P} = -< -\frac{\overline{P'}}{Q'} > \partial_Q^Q$$

Proof. Let $d \in \mathbb{N}$ be an integer. We will say that Axiom **(B4)** holds in degree $\leq d$ if for any field $F \in \mathcal{F}_k$, any irreducible primitive polynomial $P \in \mathcal{O}_v[T]$ of degree $\leq d$, any monic irreducible $Q \in \kappa(v)[T]$ then: if Q doesn't lie in the divisor D_P, the homomorphism ∂_Q^P is 0 on $K_*^{MW}(F[T]/P)$ and if Q lies in D_P and that the local ring $\mathcal{O}_{\overline{y},z}$ is a discrete valuation ring with π as uniformizing element , then the homomorphism ∂_Q^P is equal to $-\partial_Q^\pi$.

We now proceed by induction on d to prove that Axiom **(B4)** holds in degree $\leq d$ for any d. For $d = 0$ this is trivial, the case $d = 1$ is also easy.

We may use Remark 2.17 to reduce to the case the residue field $\kappa(v)$ is infinite. □

We will use:

Lemma 3.28. *Let P be a primitive irreducible polynomial of degree d in $F[T]$. Let Q be a monic irreducible polynomial in $\kappa(v)[T]$.*

Assume either that $\overline{P}$ is prime to Q, or that Q divides $\overline{P}$ and that the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with uniformizing element π.

Then the elements of the form $\eta^m[\overline{G_1}]\ldots[\overline{G_n}]$, where all the G_i's are irreducible elements in $\mathcal{O}_v[T]$ of degree $< d$, such that, either G_1 is equal to π or $\overline{G_1}$ is prime to Q, and for any $i \geq 2$, $\overline{G_i}$ is prime to Q, generate $K_*^{MW}(F[T]/P)$ as a group.

Proof. First the symbols of the form $\eta^m[\overline{G_1}]\ldots[\overline{G_n}]$ with the G_i irreducible elements of degree $< d$ of $\mathcal{O}_v[T]$ generate the Milnor–Witt K-theory of $f[T]/P$ as a group.

1) We first assume that $\overline{P}$ is prime to Q. It suffices to check that those element above are expressible in terms of symbols of the form of the Lemma. Pick up one such $\eta^m[\overline{G_1}]\ldots[\overline{G_n}]$. Assume that there exists i such that $\overline{G_i}$ is divisible by Q (otherwise there is nothing to prove), for instance G_1.

If the field $\kappa(v)$ is infinite, which we may assume by Remark 2.17, we may find an $\alpha \in \mathcal{O}_v$ such that $G_1(\alpha)$ is a unit in $\mathcal{O}_v^\times$. Then there exists a unit u in $\mathcal{O}_v^\times$ and an integer v (actually the valuation of $P(\alpha)$ at π) such that $P + u\pi^v G$ is divisible by $T - \alpha$ in $\mathcal{O}_v[T]$. Write $P + u\pi^v G_1 = (T - \alpha)H_1$. Observe that Q which divides $\overline{G_1}$ and is prime to $\overline{P}$ must be prime to both $T - \overline{\alpha}$ and $\overline{H_1}$.

Observe that $\frac{(T-\alpha)}{u\pi^v}H_1 = \frac{P}{u\pi^v} + G_1$ is the Euclidean division of $\frac{(T-\alpha)}{u\pi^v}H_1$ by P. By Lemma 3.25 one has in $K_*^{MW}(F(T))$, modulo L_{d-1}

$$\eta^m[P][G_1][G_2]\ldots[G_n] = \eta^m[P][\frac{(T-\alpha)}{u\pi^v}H_1][G_2]\ldots[G_n]$$

Because $\partial^P_{D_P}$ vanishes on L_{d-1}, applying $\partial^P_{D_P}$ to the previous congruence yields the equality in $K^{MW}_*(F[T]/P)$

$$\eta^m[\overline{G_1}]\ldots[\overline{G_n}] = \eta^m[\frac{(T-\alpha)}{u\pi^v}\overline{H_1}][G_2]\ldots[\overline{G_n}]$$

Expanding $[\frac{(T-\alpha)}{u\pi^v}\overline{H_1}]$ as $[\frac{(T-\alpha)}{u\pi^v}]+[\overline{H_1}]+\eta[\frac{(T-\alpha)}{u\pi^v}][\overline{H_1}]$ shows that we may strictly reduce the number of G_i's whose mod π reduction is divisible by Q. This proves our first claim (using the relation $[\pi][\pi]=[\pi][-1]$ we may indeed assume that only G_1 is maybe equal to π).

2) Now assume that Q divides $\overline{P}$ and that the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with uniformizing element π. By our assumption, any non-zero element in the discrete valuation ring $\mathcal{O}_{D_P,Q}=(\mathcal{O}_v[T]/P)_Q$ can be written as

$$\pi^v\frac{\overline{R}}{\overline{S}}$$

with R and S polynomials in $\mathcal{O}_v[T]$ of degree $< d$ whose mod π reduction in $\kappa(v)[T]$ is prime to Q. From this, it follows easily that the symbols of the form $\eta^m[\overline{G_1}]\ldots[\overline{G_n}]$, with the G_i's being either a polynomial in $\mathcal{O}_v[T]$ of degree $< d$ whose mod π reduction in $\kappa(v)[T]$ is prime to Q, either equal to π.

The Lemma is proven. □

Now let $d>0$ and assume the claim is proven in degrees $< d$, for all fields. Let P be a primitive irreducible polynomial of degree d in $\mathcal{O}_v[T]$. Let Q be a monic irreducible polynomial in $\kappa(v)[T]$.

Under our inductive assumption, we may compute $\partial^{\pi,P}_Q(\eta^m[G_1]\ldots[\overline{G_n}])$ for any sequence G_1, .., G_n as in the Lemma.

Indeed, the symbol $\eta^m[P][G_1]\ldots[\overline{G_n}]\in K^{MW}_{n-m}$ has residue at P the symbol $\eta^m[\overline{G_1}]\ldots[\overline{G_n}]$. All its other potentially non trivial residues concern irreducible polynomials of degree $< d$. By the (proof of) Theorem 3.24, we know that there exists an $\alpha\in L_{d-1}(K^{MW}_{n-m}(F(T))$ such that

$$\eta^m[P][G_1]\ldots[\overline{G_n}]+\alpha$$

has only one non vanishing residue, which is at P, and which equals $\eta^m[\overline{G_1}]\ldots[\overline{G_n}]$.

Then the support of α (which means the set of points of codimension one in $\mathbb{A}^1_F$ where α has a non trivial residue) consists of the divisors defined by the G_i's (P doesn't appear). But those don't contain Q.

Using the commutative diagram which defines the ∂_Q^P's, we may compute $\partial_Q^{\pi,P}(\eta^m[\overline{G_1}]\dots[\overline{G_n}])$ as

$$\partial_Q^Q(\partial_v^\pi(\eta^m[P][G_1]\dots[G_n]+\alpha)) = \partial_Q^Q(\partial_v^\pi(\eta^m[P][G_1]\dots[G_n]) + \sum_i \partial_Q^{\pi,G_i}(\partial_{D_{G_i}}^{G_i}(\alpha))$$

By our inductive assumption, $\sum_i \partial_Q^{\pi,G_i}(\partial_{D_{G_i}}^{G_i}(\alpha)) = 0$ because the supports G_i do not contain Q.

We then have two cases:

(1) G_1 is not π. Then

$$\partial_v^\pi(\eta^m[P][G_1]\dots[G_n]) = 0$$

as every element lies in $\mathcal{O}_{v[T]}^\times$. Thus in that case, $\partial_Q^{\pi,P}(\eta^m[\overline{G_1}]\dots[\overline{G_n}]) = 0$ which is compatible with our claim.

(2) $G_1 = \pi$. Then

$$\begin{aligned}\partial_v^\pi(\eta^m[P][\pi][G_2]\dots[G_n]) &= - < -1 > \partial_v^\pi(\eta^m[\pi][P][G_2]\dots[G_n]) \\ &= - < -1 > \eta^m[\overline{P}][\overline{G_2}]\dots[\overline{G_n}]\end{aligned}$$

Applying ∂_Q^Q yields 0 if $\overline{P}$ is prime to Q, as all the terms are units. If $\overline{P} = QR$, then R is a unit in $(\mathbb{A}^1_{\kappa v})_Q$ by our assumptions. Expending $[QR] = [Q]+[R]+\eta[Q][R]$, we get

$$\begin{aligned}\partial_Q^{\pi,P}(\eta^m[\overline{G_1}]\dots[\overline{G_n}]) &= - < -1 > \eta^m([\overline{G_2}]\dots[\overline{G_n}] + \eta[\overline{R}][\overline{G_2}]\dots[\overline{G_n}]) \\ &= - < -\overline{R} > \eta^m[\overline{G_2}]\dots[\overline{G_n}]\end{aligned}$$

It remains to observe that $\overline{R} = \frac{\overline{P'}}{Q'}$.

By the previous Lemma the symbols we used generate $K_*^{MW}(F[T]/P)$. Thus the previous computations prove the Theorem. □

Now we want to prove Axiom **(B5)**. Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Choose a uniformizing element π of $\mathcal{O}_{X,y_0}$. Denote by $\mathcal{K}_n(X; y_0)$ the kernel of the map

$$K_n^{MW}(F) \xrightarrow{\Sigma_{y\in X^{(1)}-\{y_0\}}\partial_y} \oplus_{y\in X^{(1)}-\{y_0\}} H^1_y(X; \underline{\mathbf{K}}_n^{MW}) \tag{3.6}$$

By definition $\underline{\mathbf{K}}_n^{MW}(X) \subset \mathcal{K}_n(X; y_0)$. The morphism $\partial_{y_0}^\pi : K_n^{MW}(F) \to K_{n-1}^{MW}(\kappa(y_0))$ induces an injective homomorphism $\mathcal{K}_n(X; y_0)/\underline{\mathbf{K}}_n^{MW}(X) \subset K_{n-1}^{MW}(\kappa(y_0))$.

We first observe:

Lemma 3.29. *Keep the previous notations and assumptions. Then* $\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \subset \mathcal{K}_n(X; y_0)/\underline{\mathbf{K}}_n^{MW}(X) \subset K_{n-1}^{MW}(\kappa(y_0))$.

Proof. We apply Gabber's lemma to y_0, and in this way, we see (by diagram chase) that we can reduce to the case $X = (\mathbb{A}_U^1)_z$ where U is a smooth local k-scheme of dimension 1. As Theorem 3.27 implies Axiom **(B4)**, we know by Lemma 2.43 that the following complex

$$0 \to \underline{\mathbf{K}}_n^{MW}(X) \to K_n^{MW}(F) \xrightarrow{\Sigma_{y \in X^{(1)}} \partial_y} \oplus_{y \in X^{(1)}} H_y^1(X; \underline{\mathbf{K}}_n^{MW}) \to H_z^2(X; \underline{\mathbf{K}}_n^{MW}) \to 0$$

is an exact sequence. Moreover, we know also from there that for $\overline{y}_0$ smooth, the morphism $H_y^1(X; \underline{\mathbf{K}}_n^{MW}) \to H_z^2(X; \underline{\mathbf{K}}_n^{MW})$ can be "interpreted" as the residue map. Its kernel is thus $\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \subset K_{n-1}^{MW}(\kappa(y_0)) \cong H_y^1(X; \underline{\mathbf{K}}_n^{MW})$. The exactness of the previous complex implies that

$$\mathcal{K}_n(X; y_0)/\underline{\mathbf{K}}_n^{MW}(X) = \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$$

proving the statement. □

Our last objective is now to show that in fact $\underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) = \mathcal{K}_n(X; y_0)/\underline{\mathbf{K}}_n^{MW}(X) \subset K_{n-1}^{MW}(\kappa(y_0))$. To do this we observe that by Lemma 2.43, for k infinite, the morphism (3.6) above is an epimorphism. Thus the previous statement is equivalent to the fact that the diagram

$$0 \to \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \to K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) \xrightarrow{\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_y} \oplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; \underline{\mathbf{K}}_n^{MW}) \to 0$$

is a short exact sequence or in other words that the epimorphism

$$\Phi_n(X; y_0) : K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0}) \xrightarrow{\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_y} \oplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; \underline{\mathbf{K}}_n^{MW}) \tag{3.7}$$

is an isomorphism. We also observe that the group $K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$ doesn't depend actually on the choice of a local parametrization of $\overline{y_0}$.

Theorem 3.30. *Let X be a local smooth k-scheme of dimension 2, with field of functions F and closed point z, let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k. Then the epimorphism $\Phi_n(X; y_0)$(3.7) is an isomorphism.*

Proof. We know from Axiom **(B1)** (that is to say Theorem 3.27) and Lemma 2.43 that the assertion is true for X a localization of $\mathbb{A}^1_U$ at some codimension 2 point, where U is a smooth local k-scheme of dimension 1. □

Lemma 3.31. *Given any element $\alpha \in K_n^{MW}(F)$, write it as $\alpha = \sum_i \alpha_i$, where the α_i's are pure symbols. Let $Y \subset X$ be the union of the hypersurfaces defined by each factor of each pure symbol α_i. Let $X \to \mathbb{A}^1_U$ be an étale morphism with U smooth local of dimension 1, with field of functions E, such that $Y \to \mathbb{A}^1_U$ is a closed immersion. Then for each i there exists a pure symbol $\beta_i \in K_n^{MW}(E(T))$ which maps to α_i modulo $\underline{\mathbf{K}}_n^{MW}(X) \subset K_n^{MW}(F)$.*

As a consequence, if $\partial_y(\alpha) \neq 0$ in $H^1_y(X; \underline{\mathbf{K}}_n^{MW})$ for some $y \in X^{(1)}$ then $y \in Y$ and $\partial_y(\alpha) = \partial_y(\beta) =\in H^1_y(X; \underline{\mathbf{K}}_n^{MW}) = H^1_y(\mathbb{A}^1_U; \underline{\mathbf{K}}_n^{MW})$.

Proof. Let us denote by π_j the irreducible elements in the factorial ring $\mathcal{O}(U)[T]$ corresponding to the irreducible components of $Y \subset \mathbb{A}^1_U$. Each $\alpha_i = [\alpha_i^1] \dots [\alpha_i^n]$ is a pure symbol in which each term α_i^s decomposes as a product $\alpha_i^s = u_i^s {\alpha'}_i^s$ of a unit u_i^s in $\mathcal{O}(X)^\times$ and a product ${\alpha'}_i^s$ of π_j's (this follows from our choices and the factoriality property of $A := \mathcal{O}(X)$. Thus α'_i is in the image of $K_n^{MW}(E(T)) \to K_n^{MW}(F)$. Now by construction, $A/(\Pi\pi_j) = B/(\Pi\pi_j)$, where $B = \mathcal{O}(U)[T]$. Thus one may choose unit v_i^s in $B^\times$ with $w_i^s := \frac{u_i^s}{v_i^s} \equiv 1[\Pi\pi_j]$.

Now set $\beta_i^s = v_i^s {\alpha'}_i^s$, $\beta_i := [\beta_i^1] \dots [\beta_i^n]$. Then we claim that β_i maps to α_i modulo $\underline{\mathbf{K}}_n^{MW}(X) \subset K_n^{MW}(F)$. In other words, we claim that $[\alpha_i^1] \dots [\alpha_i^n] - [\beta_i^1] \dots [\beta_i^n]$ lies in $\underline{\mathbf{K}}_n^{MW}(X)$ which means that each of its residue at any point of codimension one in X vanishes. Clearly, by construction the only non-zero residues can only occur at each π_j.

We end up in showing the following: given elements $\beta^s \in A - \{0\}$, $s \in \{1, \dots, n\}$ and $w^s \in A^\times$ which is congruent to 1 modulo each irreducible element π which divides one of the β^s, then for each such π, $\partial^\pi([\beta^1] \dots [\beta^n]) = \partial^\pi([w^1\beta^1] \dots [w^n\beta^n])$. We expand $[w^1\beta^1] \dots [w^n\beta^n]$ as $[w^1][w^2\beta^2] \dots [w^n\beta^n] + [\beta^1][w^2\beta^2] \dots [w^n\beta^n] + \eta[w^1][\beta^1][w^2\beta^2] \dots [w^n\beta^n]$. Now using Proposition 3.17 and the fact that $\overline{w^i}^\pi = 1$, we immediately get $\partial^\pi([w^1\beta^1] \dots [w^n\beta^n]) = \partial^\pi([\beta^1][w^2\beta^2] \dots [w^n\beta^n])$ which gives the result. An easy induction gives the result. This proof can obviously be adapted for pure symbols of the form $\eta^n[\alpha]$. □

Now the theorem follows from the Lemma. Let $\overline{\alpha} \in K_n^{MW}(F)/\underline{\mathbf{K}}_n^{MW}(X) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$ be in the kernel of $\Phi_n(X; y_0)$. Assume $\alpha \in K_n^{MW}(F)$ represents $\overline{\alpha}$. By Gabber's Lemma there exists an étale morphism $X \to \mathbb{A}^1_U$ with U smooth local of dimension 1, with field of functions E, such that $Y \cup \overline{y_0} \to \mathbb{A}^1_U$ is a closed immersion, where Y is obtained by writing α as a sum of pure symbols α_i's. By the previous Lemma, we may find β_i in $K_n^{MW}(E(T))$ mapping to α modulo $\underline{\mathbf{K}}_n^{MW}(X)$ yo α_i. Let β be the sum of the β_i's. Then $\overline{\beta} \in K_n^{MW}(E(T))/\underline{\mathbf{K}}_n^{MW}((\mathbb{A}^1_U)_z) + \underline{\mathbf{K}}_{n-1}^{MW}(\mathcal{O}_{y_0})$ is also in the kernel of our morphism $\Phi_n((\mathbb{A}^1_U)_z; y_0)$. Thus $\overline{\beta} = 0$ and so $\overline{\alpha} = 0$. □

Unramified $K^{\mathcal{R}}$-theories. We now slightly generalize our construction by allowing some "admissible" relations in $K_*^{MW}(F)$. An admissible set of relations $\mathcal{R}$ is the datum for each $F \in \mathcal{F}_k$ of a graded ideal $\mathcal{R}_*(F) \subset K_*^{MW}(F)$ with the following properties:

1. For any extension $E \subset F$ in $\mathcal{F}_k$, $\mathcal{R}_*(E)$ is mapped into $\mathcal{R}_*(F)$.
2. For any discrete valuation v on $F \in \mathcal{F}_k$, any uniformizing element π, $\partial_v^\pi(\mathcal{R}_*(F)) \subset \mathcal{R}_*(\kappa(v))$.
3. For any $F \in \mathcal{F}_k$ the following sequence is a short exact sequence:

$$0 \to \mathcal{R}_*(F) \to \mathcal{R}_*(F(T)) \xrightarrow{\sum_P \partial^P_{D_P}} \oplus_P \mathcal{R}_{*-1}(F[t]/P) \to 0 \qquad \square$$

The third one is usually more difficult to check.

Given an admissible relation $\mathcal{R}$, for each $F \in \mathcal{F}_k$ we simply denote by $K_*^{\mathcal{R}}(F)$ the quotient graded ring $K_*^{MW}(F)/\mathcal{R}_*(F)$. The property (1) above means that we get this way a functor

$$\mathcal{F}_k \to \mathcal{A}b_*$$

This functor is moreover endowed with data **(D4) (i)** and **(D4) (ii)** coming from the K_*^{MW}-algebra structure. The property (2) defines the data **(D4) (iii)**. The axioms **(B0)**, **(B1)**, **(B2)**, **(B3)** are immediate consequences from those for K_*^{MW}. Property (3) implies axiom **(HA) (i)**. Axiom **(HA) (ii)** is clear. Axioms **(B4)** and **(B5)** are also consequences from the corresponding axioms just established for K_*^{MW}. We thus get as in Theorem 2.46 a $\mathbb{Z}$-graded strongly $\mathbb{A}^1$-invariant sheaf, denoted by $\underline{\mathbf{K}}_*^{\mathcal{R}}$ with isomorphisms $(\underline{\mathbf{K}}_n^{\mathcal{R}})_{-1} \cong \underline{\mathbf{K}}_{n-1}^{\mathcal{R}}$. There is obviously a structure of $\mathbb{Z}$-graded sheaf of algebras over $\underline{\mathbf{K}}_*^{MW}$.

Lemma 3.32. *Let $R_* \subset K_*^{MW}(k)$ be a graded ideal. For any $F \in \mathcal{F}_k$, denote by $\mathcal{R}_*(F) := R_*.K_*^{MW}(F)$ the ideal generated by R_*. Then $\mathcal{R}_*(F)$ is an admissible relation on K_*^{MW}. We denote the quotient simply by $K_*^{MW}(F)/R_*$.*

Proof. Properties (1) and (2) are easy to check. We claim that the property (3) also hold: this follows from Theorem 3.24 which states that the morphisms and maps are $K_*^{MW}(F)$-module morphisms. $\square$

Of course when $R_* = 0$, we get the $\mathbb{Z}$-graded sheaf of unramified Milnor–Witt K-theory

$$\underline{\mathbf{K}}_*^{MW}$$

itself.

Example 3.33. For instance we may take an integer n and $R_* = (n) \subset K_*^{MW}(k)$; we obtain mod n Milnor–Witt unramified sheaves. For $R_* = (\eta)$ the ideal generated by η, this yields unramified Milnor K-theory $\underline{\mathbf{K}}_*^M$. For

$R_* = (n, \eta)$ this yields mod n Milnor K-theory. For $\mathcal{R} = (h)$, this yields Witt K-theory $\underline{\mathbf{K}}_*^W$, for $\mathcal{R} = (\eta, \ell)$ this yields mod ℓ Milnor K-theory. □

Example 3.34. Let $\mathcal{R}_*^I(F)$ be the kernel of the epimorphism $K_*^{MW}(F) \twoheadrightarrow I^*(F)$, $[u] \mapsto < u > -1 = - << u >>$ described in [54], see also Remark 3.12. Then $\mathcal{R}_*^I(F)$ is admissible. Recall from the Remark 3.12 that $K_*^{MW}(F)[\eta^{-1}] = W(F)[\eta, \eta^{-1}]$ and that $I^*(F)$ is the image of $K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$. Now the morphism $K_*^{MW}(F) \to W(F)[\eta, \eta^{-1}]$ commutes to every data. We conclude using the Lemma 4.5 below. Thus we get in this way unramified sheaves of powers of the fundamental ideal $\underline{\mathbf{I}}^*$ (see also [53]). □

Let $\phi : M_* \to N_*$ be a morphism (in the obvious sense) of between functors $\mathcal{F}_k \to \mathcal{A}b_*$ endowed with data **(D4) (i)**, **(D4) (ii)** and **(D4) (iii)** and satisfying the Axioms **(B0)**, **(B1)**, **(B2)**, **(B3)**, **(HA)**, **(B4)** and **(B5)** of Theorem 2.46.

Denote for each $F \in \mathcal{F}_k$ by $Im(\phi)_*(F)$ (resp. $Ker(\phi)_*(F)$) the image (resp. the kernel) of $\phi(F) : M_*(F) \to N_*(F)$. One may extend both to functor $\mathcal{F}_k \to \mathcal{A}b_*$ with data **(D4) (i)**, **(D4) (ii)** and **(D4) (iii)** induced from the one on M_* and N_*.

Lemma 3.35. *Let $\phi : M_* \to N_*$ be a morphism of as above. Then $Im(\phi)_*$ and $Ker(\phi)_*$ with the induced Data* **(D4) (i)**, **(D4) (ii)** *and* **(D4) (iii)** *satisfy the Axioms* **(B0)**, **(B1)**, **(B2)**, **(B3)**, **(HA)**, **(B4)** *and* **(B5)** *of Theorem 2.46.*

Proof. The only difficulty is to check axiom **(HA) (i)**. It is in fact very easy to check it using the axioms **(HA) (i)** and **(HA) (ii)** for M_* and N_*. Indeed **(HA) (ii)** provides a splitting of the short exact sequences of **(HA) (i)** for M_* and N_* which are compatible. One gets the axiom **(HA) (i)** for $Im(\phi)_*$ and $Ker(\phi)_*$ using the snake lemma. We leave the details to the reader. □

3.3 Milnor–Witt K-Theory and Strongly $\mathbb{A}^1$-Invariant Sheaves

Fix a natural number $n \geq 1$. Recall from [59] that $(\mathbb{G}_m)^{\wedge n}$ denotes the n-th smash power of the pointed space $\mathbb{G}_m$. We first construct a canonical morphism of pointed spaces

$$\sigma_n : (\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$$

$(\mathbb{G}_m)^{\wedge n}$ is *a priori* the associated sheaf to the naive presheaf $\Theta_n : X \mapsto (\mathcal{O}^\times(X))^{\wedge n}$ but in fact:

Lemma 3.36. *The presheaf $\Theta_n : X \mapsto (\mathcal{O}(X)^\times)^{\wedge n}$ is an unramified sheaf of pointed sets.*

Proof. It is as a presheaf unramified in the sense of our Definition 2.1 thus automatically a sheaf in the Zariski topology. One may check it is a sheaf in the Nisnevich topology by checking Axiom **(A1)**. One has only to use the following observation: let E_α be a family of pointed subsets in a pointed set E. Then $\cap_\alpha (E_\alpha)^{\wedge n} = (\cap_\alpha E_\alpha)^{\wedge n}$, where the intersection is computed inside $E^{\wedge n}$. □

Fix an irreducible $X \in Sm_k$ with function field F. There is a tautological symbol map $(\mathcal{O}(X)^\times)^{\wedge n} \subset (F^\times)^{\wedge n} \to K_n^{MW}(F)$ that takes a symbol $(u_1, \ldots, u_n) \in (\mathcal{O}(X)^\times)^{\wedge n}$ to the corresponding symbol in $[u_1]\ldots[u_n] \in K_n^{MW}(F)$. But this symbol $[u_1]\ldots[u_n] \in K_n^{MW}(F)$ lies in $\underline{\mathbf{K}}_n^{MW}(X)$, that is to say each of its residues at points of codimension 1 in X is 0. This follows at once from the definitions and elementary formulas for the residues.

This defines a morphism of sheaves on $\tilde{Sm}_k$. Now to show that this extends to a morphism of sheaves on Sm_k, using the equivalence of categories of Theorem 2.11 (and its proof) we end up to show that our symbol maps commutes to restriction maps s_v, which is also clear from the elementary formulas we proved in Milnor–Witt K-theory. In this way we have obtained our canonical symbol map

$$\sigma_n : (\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$$

From what we have done in Chaps. 2 and 3, we know that $\underline{\mathbf{K}}_n^{MW}$ is a strongly $\mathbb{A}^1$-invariant sheaf.

Theorem 3.37. *Let $n \geq 1$. The morphism σ_n is the universal morphism from $(\mathbb{G}_m)^{\wedge n}$ to a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. In other words, given a morphism of pointed sheaves $\phi : (\mathbb{G}_m)^{\wedge n} \to M$, with M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, then there exists a unique morphism of sheaves of abelian groups $\Phi : \underline{\mathbf{K}}_n^{MW} \to M$ such that $\Phi \circ \sigma_n = \phi$.*

Remark 3.38. The statement is wrong if we release the assumption that M is a sheaf of abelian groups. The free strongly $\mathbb{A}^1$-invariant sheaf of groups generated by $\mathbb{G}_m$ will be seen in Sect. 7.3 to be non commutative. For $n = 2$, it is a sheaf of abelian groups. For $n > 2$ it is not known to us.

The statement is also false for $n = 0$: $(\mathbb{G}_m)^{\wedge 0}$ is just $Spec(k)_+$, that is to say $Spec(k)$ with a base point added, and the free strongly $\mathbb{A}^1$-invariant sheaf of abelian groups generated by $Spec(k)_+$ is $\mathbb{Z}$, not $\underline{\mathbf{K}}_0^{MW}$. To see a analogous presentation of $\underline{\mathbf{K}}_0^{MW}$ see Theorem 3.46 below. □

Roughly, the idea of the proof is to first use Lemma 3.4 to show that $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ induces on fields $F \in \mathcal{F}_k$ a morphism $K_n^{MW}(F) \to M(F)$ and then to use our work on unramified sheaves in Chap. 2 to observe this induces a morphism of sheaves.

Theorem 3.39. *Let M be a strongly $\mathbb{A}^1$-invariant sheaf, let $n \geq 1$ be an integer, and let $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ be a morphism of pointed sheaves. For any* *field $F \in \mathcal{F}_k$, there is unique morphism*

$$\Phi(F) : K_n^{MW}(F) \to M(F)$$

such that for any $(u_1, \ldots, u_n) \in (F^\times)^n$, $\Phi_n(F)([u_1, \ldots, u_n]) = \phi(u_1, \ldots, u_n)$.

Preliminaries. We will freely use some notions and some elementary results from [59].

Let M be a sheaf of groups on Sm_k. Recall that we denote by M_{-1} the sheaf $M^{(\mathbb{G}_m)}$, and for $n \geq 0$, by M_{-n} the n-th iteration of this construction. To say that M is strongly $\mathbb{A}^1$-invariant is equivalent to the fact that $K(M, 1)$ is $\mathbb{A}^1$-local [59]. Indeed from *loc. cit.*, for any pointed space $\mathcal{X}$, we have $Hom_{\mathcal{H}_\bullet(k)}(\mathcal{X}; K(M, 1)) \cong H^1(\mathcal{X}; M)$ and $Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathcal{X}); K(M, 1)) \cong \tilde{M}(X))$. Here we denote for M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups and $\mathcal{X}$ a pointed space by $\tilde{M}(\mathcal{X})$ the kernel of the evaluation at the base point of $M(\mathcal{X}) \to M(k)$, so that $M(\mathcal{X})$ splits as $M(k) \oplus \tilde{M}(\mathcal{X})$.

We also observe that because M is assumed to be abelian, the map (from "pointed to base point free classes")

$$Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathcal{X}); K(M, 1)) \to Hom_{\mathcal{H}(k)}(\Sigma(\mathcal{X}); K(M, 1))$$

is a bijection.

From Lemma 2.32 and its proof we know that in that case, $R\mathbf{Hom}_\bullet(\mathbb{G}_m; K(M, 1))$ is canonically isomorphic to $K(M_{-1}, 1)$ and that M_{-1} is also strongly $\mathbb{A}^1$-invariant. We also know that $R\Omega_s(K(M, 1) \cong M$.

As a consequence, for a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups M, the evaluation map

$$Hom_{\mathcal{H}_\bullet(k)}(\Sigma((\mathbb{G}_m)^{\wedge n}), K(M, 1)) \to M_{-n}(k)$$

is an isomorphism of abelian groups.

Now for $\mathcal{X}$ and $\mathcal{Y}$ pointed spaces, the cofibration sequence $\mathcal{X} \vee \mathcal{Y} \to \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \wedge \mathcal{Y}$ splits after applying the suspension functor Σ. Indeed, as $\Sigma(\mathcal{X} \times \mathcal{Y})$ is a co-group object in $\mathcal{H}_\bullet(k)$ the (ordered) sum of the two morphism $\Sigma(\mathcal{X} \times \mathcal{Y}) \to \Sigma(\mathcal{X}) \vee \Sigma(\mathcal{Y}) = \Sigma(\mathcal{X} \vee \mathcal{Y})$ gives a left inverse to $\Sigma(\mathcal{X}) \vee \Sigma(\mathcal{Y}) \to \Sigma(\mathcal{X} \times \mathcal{Y})$. This left inverse determines an $\mathcal{H}_\bullet(k)$-isomorphism $\Sigma(\mathcal{X}) \vee \Sigma(\mathcal{Y}) \vee \Sigma(\mathcal{X} \wedge \mathcal{Y}) \cong \Sigma(\mathcal{X} \times \mathcal{Y})$.

We thus get canonical isomorphisms:

$$\tilde{M}(\mathcal{X} \times \mathcal{Y}) = \tilde{M}(\mathcal{X}) \oplus \tilde{M}(\mathcal{Y}) \oplus \tilde{M}(\mathcal{X} \wedge \mathcal{Y})$$

and analogously

$$H^1(\mathcal{X} \times \mathcal{Y}; M) = H^1(\mathcal{X}; M) \oplus H^1(\mathcal{Y}; M) \oplus H^1(\mathcal{X} \wedge \mathcal{Y}; M)$$

As a consequence, the product $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ on $\mathbb{G}_m$ induces in $\mathcal{H}_\bullet(k)$ a morphism $\Sigma(\mathbb{G}_m \times \mathbb{G}_m) \to \mathbb{Z}(\mathbb{G}_m)$ which using the above splitting decomposes as

$$\Sigma(\mu) = \langle Id_{\Sigma(\mathbb{G}_m)}, d_{\Sigma(\mathbb{G}_m)}, \eta \rangle : \Sigma(\mathbb{G}_m) \vee \Sigma(\mathbb{G}_m) \vee \Sigma((\mathbb{G}_m)^{\wedge 2}) \to \Sigma(\mathbb{G}_m)$$

The morphism $\Sigma((\mathbb{G}_m)^{\wedge 2}) \to \Sigma(\mathbb{G}_m)$ so defined is denoted η. It can be shown to be isomorphic in $\mathcal{H}_\bullet(k)$ to the Hopf map $\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$.

Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. We will denote by

$$\eta : M_{-2} \to M_{-1}$$

the morphism of strongly $\mathbb{A}^1$-invariant sheaves of abelian groups induced by η.

In the same way let $\Psi : \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \cong \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ be the twist morphism and for M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, we still denote by

$$\Psi : M_{-2} \to M_{-2}$$

the morphism of strongly $\mathbb{A}^1$-invariant sheaves of abelian groups induced by Ψ.

Lemma 3.40. *Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. Then the morphisms $\eta \circ \Psi$ and η*

$$M_{-2} \to M_{-1}$$

are equal.

Proof. This is a direct consequence of the fact that μ is commutative. □

As a consequence, for any $m \geq 1$, the morphisms of the form

$$M_{-m-1} \to M_{-1}$$

obtained by composing m times morphisms induced by η doesn't depend on the chosen ordering. We thus simply denote by $\eta^m : M_{-m-1} \to M_{-1}$ this canonical morphism.

Proof of Theorem 3.39 By Lemma 3.6 1), the uniqueness is clear. By a base change argument analogous to [52, Corollary 5.2.7], we may reduce to the case $F = k$.

From now on we fix a morphism of pointed sheaves $\phi : (\mathbb{G}_m)^{\wedge n} \to M$, with M a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. We first observe that ϕ determines and is determined by the $\mathcal{H}_\bullet(k)$-morphism $\phi : \Sigma((\mathbb{G}_m)^{\wedge n}) \to K(M, 1)$, or equivalently by the associated element $\phi \in M_{-n}(k)$.

For any symbol $(u_1, \dots, u_r) \in (k^\times)^r$, $r \in \mathbb{N}$, we let $S^0 \to (\mathbb{G}_m)^{\wedge r}$ be the (ordered) smash-product of the morphisms $[u_i] : S^0 \to \mathbb{G}_m$ determined by u_i. For any integer $m \geq 0$ such that $r = n + m$, we denote by $[\eta^m, u_1, \dots, u_r] \in M(k) \cong Hom_{\mathcal{H}_\bullet(k)}(\Sigma(S^0), K(M,1))$ the composition

$$\eta^m \circ \Sigma([u_1, \dots, u_n]) : \Sigma(S^0) \to \Sigma((\mathbb{G}_m)^{\wedge r}) \xrightarrow{\eta^m} \Sigma((\mathbb{G}_m)^{\wedge n}) \xrightarrow{\phi} K(M,1)$$

The theorem now follows from the following:

Lemma 3.41. *The previous assignment* $(m, u_1, \dots, u_r) \mapsto [\eta^m, u_1, \dots, u_r] \in M(k)$ *satisfies the relations of Definition 3.3 and as a consequence induce a morphism*

$$\Phi(k) : K_n^{MW}(k) \to M(k)$$

Proof. The proof of the Steinberg relation $\mathbf{1_n}$ will use the following stronger result by P. Hu and I. Kriz:

Lemma 3.42. *(Hu–Kriz [33]) The canonical morphism of pointed sheaves* $(\mathbb{A}^1 - \{0,1\})_+ \to \mathbb{G}_m \wedge \mathbb{G}_m$, $x \mapsto (x, 1-x)$ *induces a trivial morphism* $\tilde{\Sigma}(\mathbb{A}^1 - \{0,1\}) \to \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ *(where* $\tilde{\Sigma}$ *means unreduced suspension*[1]*) in* $\mathcal{H}_\bullet(k)$.

For any $a \in k^\times - \{1\}$ the suspension of the morphism of the form $[a, 1-a] : S^0 \to (\mathbb{G}_m)^{\wedge 2}$ factors in $\mathcal{H}_\bullet(k)$) through $\tilde{\Sigma}(\mathbb{A}^1 - \{0,1\}) \to \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$ as the morphism $Spec(k) \to \mathbb{G}_m \wedge \mathbb{G}_m$ factors itself through $\mathbb{A}^1 - \{0,1\}$. This implies the Steinberg relation in our context as the morphism of the form $\Sigma([u_i, 1-u_i]) : \Sigma(S^0) \to \Sigma((\mathbb{G}_m)^{\wedge 2})$ appears as a factor in the morphism which defines the symbol $[\eta^m, u_1, \dots, u_r]$, with $u_i + u_{i+1} = 1$, in $M(k)$.

Now, to check the relation $\mathbf{2}_n$, we observe that the pointed morphism $[ab] : S^0 \to \mathbb{G}_m$ factors as $S^0 \xrightarrow{[a][b]} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m$. Taking the suspension and using the above splitting which defines η, yields that

$$\Sigma([ab]) = \Sigma([a]) \vee \Sigma([b]) \vee \eta([a][b]) : \Sigma(S^0) \to \Sigma(\mathbb{G}_m)$$

in the group $Hom_{\mathcal{H}_\bullet(k)}(\Sigma(S^0), \Sigma(\mathbb{G}_m))$ whose law is denoted by $\vee$. This implies relation $\mathbf{2}_n$.

Now we come to check the relation $\mathbf{4}_n$. For any $a \in k^\times$, the morphism $a : \mathbb{G}_m \to \mathbb{G}_m$ given by multiplication by a is not pointed (unless $a = 1$). However the pointed morphism $a_+ : (\mathbb{G}_m)_+ \to \mathbb{G}_m$ induces after suspension $\Sigma(a_+) : S^1 \vee \Sigma(\mathbb{G}_m) \cong \Sigma((\mathbb{G}_m)_+) \to \Sigma(\mathbb{G}_m)$. We denote by $< a >: \Sigma(\mathbb{G}_m) \to \Sigma(\mathbb{G}_m)$ the morphism in $\mathcal{H}_\bullet(k)$ induced on the factor $\Sigma(\mathbb{G}_m)$. We need:

---

[1] Observe that if $k = \mathbb{F}_2$, $\mathbb{A}^1 - \{0,1\}$ has no rational point.

Lemma 3.43. *1) For any $a \in k^\times$, the morphism $M_{-1} \to M_{-1}$ induced by $< a >: \Sigma(\mathbb{G}_m) \to \Sigma(\mathbb{G}_m)$ is equal to $Id + \eta \circ [a]$.*

2) The twist morphism $\Psi \in Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$ and the inverse, for the group structure, of $Id_{\mathbb{G}_m} \wedge < -1 > \cong < -1 > \wedge Id_{\mathbb{G}_m}$ have the same image in the set $Hom_{\mathcal{H}(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$.

Remark 3.44. In fact the map

$$Hom_{\mathcal{H}_\bullet(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)) \to Hom_{\mathcal{H}(k)}(\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m), \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$$

is a bijection. Indeed we know that $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m))$ is $\mathbb{A}^1$-equivalent to $\mathbb{A}^2 - \{0\}$ and also to SL_2 because the morphism $SL_2 \to \mathbb{A}^2 - \{0\}$ (forgetting the second column) is an $\mathbb{A}^1$-weak equivalence. As SL_2 is a group scheme, the classical argument shows that this space is $\mathbb{A}^1$-simple. Thus for any pointed space $\mathcal{X}$, the action of $\pi_1^{\mathbb{A}^1}(SL_2)(k)$ on $Hom_{\mathcal{H}_\bullet(k)}(\mathcal{X}, SL_2)$ is trivial. We conclude because as usual, for any pointed spaces $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{Y}$ $\mathbb{A}^1$-connected, the map $Hom_{\mathcal{H}_\bullet(k)}(\mathcal{X}, \mathcal{Y}) \to Hom_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y})$ is the quotient by the action of the group $\pi_1^{\mathbb{A}^1}(\mathcal{Y})(k)$.

Proof. 1) The morphism $a : \mathbb{G}_m \to \mathbb{G}_m$ is equal to the composition $\mathbb{G}_m \overset{[a]\times Id}{\to} \mathbb{G}_m \times \mathbb{G}_m \overset{\mu}{\to} \mathbb{G}_m$. Taking the suspension, the previous splittings give easily the result.

2) Through the $\mathcal{H}_\bullet(k)$-isomorphism $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \cong \mathbb{A}^2 - \{0\}$, the twist morphism becomes the opposite of the permutation isomorphism $(x, y) \mapsto (y, x)$. This follows easily from the definition of this isomorphism using the Mayer–Vietoris square

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \subset & \mathbb{A}^1 \times \mathbb{G}_m \\ \cap & & \cap \\ \mathbb{G}_m \times \mathbb{A}^1 & \subset & \mathbb{A}^2 - \{0\} \end{array}$$

and the fact that our automorphism on $\mathbb{A}^2 - \{0\}$ permutes the top right and bottom left corner.

Consider the action of $GL_2(k)$ on $\mathbb{A}^2 - \{0\}$. As any matrix in $SL_2(k)$ is a product of elementary matrices, the associated automorphism $\mathbb{A}^2 - \{0\} \cong \mathbb{A}^2 - \{0\}$ is the identity in $\mathcal{H}(k)$. As the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is congruent to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ modulo $SL_2(k)$, we get the result. □

Proof of Theorem 3.37 By Lemma 3.45 below, we know that for any smooth irreducible X with function field F, the restriction map $M(X) \subset M(F)$ is injective.

As $\underline{\mathbf{K}}_n^{MW}$ is unramified, the Remark 2.15 of Sect. 1.1 shows that to produce a morphism of sheaves $\Phi : \underline{\mathbf{K}}_n^{MW} \to M$ it is sufficient to prove that for any discrete valuation v on $F \in \mathcal{F}_k$ the morphism $\Phi(F) : K_n^{MW}(F) \to M(F)$ maps $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ into $M(\mathcal{O}_v)$ and in case the residue field $\kappa(v)$ is separable, that some square is commutative (see Remark 2.15).

But by Theorem 3.22, we know that the subgroup $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ of $K_n^{MW}(F)$ is the one generated by symbols of the form $[u_1, \ldots, u_n]$, with the $u_i \in \mathcal{O}_v^\times$. The claim is now trivial: for any such symbol there is a smooth model X of $\mathcal{O}_v$ and a morphism $X \to (\mathbb{G}_m)^{\wedge n}$ which induces $[u_1, \ldots, u_n]$ when composed with $(\mathbb{G}_m)^{\wedge n} \to \underline{\mathbf{K}}_n^{MW}$. But now composition with $\phi : (\mathbb{G}_m)^{\wedge n} \to M$ gives an element of $M(X)$ which lies in $M(\mathcal{O}_v) \subset M(F)$ which is by definition the image of $[u_1, \ldots, u_n]$ through $\Phi(F)$. A similar argument applies to check the commutativity of the square of the Remark 2.15: one may choose X so that there is a closed irreducible $Y \subset X$ of codimension 1, with $\mathcal{O}_{X,\eta_Y} = \mathcal{O}_v \subset F$. Then the restriction of $\Phi([u_1, \ldots, u_n]) \subset M(\mathcal{O}_v)$ is just induced by the composition $Y \to X \to (\mathbb{G}_m)^{\wedge n} \to M$, and this is also compatible with the s_v in Milnor–Witt K-theory. □

Lemma 3.45. *Let M be an $\mathbb{A}^1$-invariant sheaf of pointed sets on Sm_k. Then for any smooth irreducible X with function field F, the kernel of the restriction map $M(X) \subset M(F)$ is trivial.*

In case M is a sheaf of groups, we see that the restriction map $M(X) \to M(F)$ is injective.

Proof. This follows from [52, Lemma 6.1.4] which states that $L_{\mathbb{A}^1}(X/U)$ is always 0-connected for U non-empty dense in X. Now the kernel of $M(X) \to M(U)$ is covered by $Hom_{\mathcal{H}_\bullet(k)}(X/U, M)$, which is trivial as M is his own π_0 and $L_{\mathbb{A}^1}(X/U)$ is 0-connected. □

We know deal with $\underline{\mathbf{K}}_0^{MW}$. We observe that there is a canonical morphism of sheaves of sets $\mathbb{G}_m/2 \to \underline{\mathbf{K}}_0^{MW}$, $U \mapsto < U >$, where $\mathbb{G}_m/2$ means the cokernel in the category of sheaves of abelian groups of $\mathbb{G}_m \xrightarrow{2} \mathbb{G}_m$.

Theorem 3.46. *The canonical morphism of sheaves $\mathbb{G}_m/2 \to \underline{\mathbf{K}}_0^{MW}$ is the universal morphism of sheaves of sets to a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. In other words $\underline{\mathbf{K}}_0^{MW}$ is the free strongly $\mathbb{A}^1$-invariant sheaf on the space $\mathbb{G}_m/2$.*

Proof. Let M be a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. Denote by $\mathbb{Z}[\mathcal{S}]$ the free sheaf of abelian groups on a sheaf of sets $\mathcal{S}$. When $\mathcal{S}$ is pointed, then the latter sheaf splits canonically as $\mathbb{Z}[\mathcal{S}] = \mathbb{Z} \oplus \mathbb{Z}(\mathcal{S})$ where $\mathbb{Z}(\mathcal{S})$ is the free sheaf of abelian groups on the pointed sheaf of sets $\mathcal{S}$, meaning the quotient $\mathbb{Z}[\mathcal{S}]/\mathbb{Z}[*]$ (where $* \to \mathcal{S}$ is the base point). Now a morphism of sheaves of sets $\mathbb{G}_m/2 \to M$ is the same as a morphism of sheaves of abelian groups $\mathbb{Z}[\mathbb{G}_m] = \mathbb{Z} \oplus \mathbb{Z}(\mathbb{G}_m) \to M$. By the Theorem 3.37 a morphism $\mathbb{Z}(\mathbb{G}_m) \to M$ is the same as a morphism $\underline{\mathbf{K}}_1^{MW} \to M$.

Thus to give a morphism of sheaves of sets $\mathbb{G}_m/2 \to M$ is the same as to give a morphism of sheaves of abelian groups $\mathbb{Z} \oplus \underline{\mathbf{K}}_1^{MW} \to M$ together with extra conditions. One of this conditions is that the composition $\mathbb{Z}\oplus\underline{\mathbf{K}}_1^{MW} \stackrel{[2]}{\to} \mathbb{Z}\oplus\underline{\mathbf{K}}_1^{MW} \to M$ is equal to $\mathbb{Z}\oplus\underline{\mathbf{K}}_1^{MW} \stackrel{[*]}{\to} \mathbb{Z}\oplus\underline{\mathbf{K}}_1^{MW} \to M$. Here $[*]$ is represented by the matrix $\begin{pmatrix} Id_{\mathbb{Z}} & 0 \\ 0 & 0 \end{pmatrix}$ and $[2]$ by the matrix $\begin{pmatrix} Id_{\mathbb{Z}} & 0 \\ 0 & [2]_1 \end{pmatrix}$. The morphism $[2]_1 : \underline{\mathbf{K}}_1^{MW} \to \underline{\mathbf{K}}_1^{MW}$ is the one induced by the square map on $\mathbb{G}_m$. From Lemma 3.14, we know that this map is the multiplication by $2_\epsilon = h$. recall that we set $\underline{\mathbf{K}}_1^{W} := \underline{\mathbf{K}}_1^{MW}/h$. Thus any morphism of sheaves of sets $\mathbb{G}_m/2 \to M$ determines a canonical morphism $\mathbb{Z}\oplus\underline{\mathbf{K}}_1^{W} \to M$. Moreover the morphism $\mathbb{Z}[\mathbb{G}_m] \to \mathbb{Z}\oplus\underline{\mathbf{K}}_1^{W}$ factors through $\mathbb{Z}[\mathbb{G}_m] \to \mathbb{Z}[\mathbb{G}_m/2]$; this morphism is induced by the map $U \mapsto (1, < U >)$. □

We have thus proven that given any morphism $\phi : \mathbb{Z}[\mathbb{G}_m/2] \to M$, there exists a unique morphism $\mathbb{Z}\oplus\underline{\mathbf{K}}_1^{W} \to M$ such that the composition $\mathbb{Z}[\mathbb{G}_m/2] \to \mathbb{Z}\oplus\underline{\mathbf{K}}_1^{W} \to M$ is ϕ. As $\mathbb{Z}\oplus\underline{\mathbf{K}}_1^{W}$ is a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, it is the free one on $\mathbb{G}_m/2$.

Our claim is now that the canonical morphism $i : \mathbb{Z}\oplus K_1^{W} \to \underline{\mathbf{K}}_0^{MW}$ is an isomorphism.

We know proceed closely to proof of Theorem 3.37. We first observe that for any $F \in \mathcal{F}_k$, the canonical map $\mathbb{Z}[F^\times/2] \to \mathbb{Z}\oplus K_1^W(F)$ factors through $\mathbb{Z}[F^\times/2] \twoheadrightarrow K_0^{MW}(F)$. This is indeed very simple to check using the presentation of $K_0^{MW}(F)$ given in Lemma 3.9. We denote by $j(F) : K_0^{MW}(F) \to \mathbb{Z}\oplus K_1^W(F)$ the morphism so obtained.

Using Theorem 3.22 and the same argument as in the end of the proof of Theorem 3.37 we see that the $j(F)$'s actually come from a morphism of sheaves $j : \underline{\mathbf{K}}_0^{MW} \to \mathbb{Z}\oplus\underline{\mathbf{K}}_1^{W}$. It is easy to check on $F \in \mathcal{F}_k$ that i and j are inverse morphisms to each other. □

The following corollary is immediate from the Theorem and its proof:

Corollary 3.47. *The canonical morphism*

$$K_1^W(F) \to I(F)$$

is an isomorphism.

We now give some applications concerning abelian sheaves of the form M_{-1}, see Sect. 2.2. From Lemma 2.32 if M is strongly $\mathbb{A}^1$-invariant, so is M_{-1}. Now we observe that there is a canonical pairing:

$$\mathbb{G}_m \times M_{-1} \to M$$

In case M is a sheaf of abelian groups, as opposed to simply a sheaf of groups, we may view $M_{-1}(X)$ for $X \in Sm_k$ as fitting in a short exact sequence:

$$0 \to M(X) \to M(\mathbb{G}_m \times X) \to M_{-1}(X) \to 0 \tag{3.8}$$

Given $\alpha \in \mathcal{O}(X)^\times$ that we view as a morphism $X \to \mathbb{G}_m$, we may consider the evaluation at α $ev_\alpha : M(\mathbb{G}_m \times X) \to M(X)$, that is to say the restriction map through $(\alpha, Id_X) \circ \Delta_X : X \to \mathbb{G}_m \times X$. Now $ev_\alpha - ev_1 : M(\mathbb{G}_m \times X) \to M(X)$ factor through $M_{-1}(X)$ and induces a morphism $\alpha\cup : M_{-1}(X) \to M(X)$. This construction define a morphism of sheaves of sets $\mathbb{G}_m \times M_{-1} \to M$ which is our pairing.

Iterating this process gives a pairing

$$(\mathbb{G}_m)^{\wedge n} \times M_{-n} \to M$$

for any $n \geq 1$.

Lemma 3.48. *For any $n \geq 1$ and any strongly $\mathbb{A}^1$-invariant sheaf, the above pairing induces a bilinear pairing*

$$\underline{\mathbf{K}}_n^{MW} \times M_{-n} \to M \quad , \quad (\alpha, m) \mapsto \alpha.m$$

Proof. Let's us prove first that for each field $F \in \mathcal{F}_k$, the pairing $(F^\times)^{\wedge n} \times M_{-n}(F) \to M(F)$ factors through $\mathbb{Z}(F^\times) \times M_{-1}(F) \to K_n^{MW}(F) \times M_{-n}(F)$. Fix $F_0 \in \mathcal{F}_k$ and consider an element $u \in M_{-n}(F_0)$. We consider the natural morphism of sheaves of abelian groups on Sm_{F_0}, $\mathbb{Z}((\mathbb{G}_m)^{\wedge n}) \to M|_{F_0}$ induced by the cup product with u, where $M|_{F_0}$ is the "restriction" of M to Sm_{F_0}. It is clearly a strongly $\mathbb{A}^1$-invariant sheaf of groups (use an argument of passage to the colimit in the H^1) and by Theorem 3.37, this morphism $\mathbb{Z}((\mathbb{G}_m)^{\wedge n}) \to M|_{F_0}$ induces a unique morphism $\underline{\mathbf{K}}_n^{MW} \to M|_{F_0}$. Now the evaluation of this morphism on F_0 itself is a homomorphism $K_n^{MW}(F_0) \to M(F_0)$ and it is induced by the product by u. This proves that the pairing $(F^\times)^{\wedge n} \times M_{-n}(F) \to M(F)$ factors through $\mathbb{Z}(F^\times) \times M_{-1}(F) \to K_n^{MW}(F) \times M_{-n}(F)$. Now to check that this comes from a morphisms of sheaves

$$\underline{\mathbf{K}}_n^{MW} \times M_{-n} \to M$$

is checked using the techniques from Sect. 2.1. The details are left to the reader. □

Now let us observe that the sheaves of the form M_{-1} are endowed with a canonical action of $\mathbb{G}_m$. We start with the short exact sequence (3.8):

$$0 \to M(X) \to M(\mathbb{G}_m \times X) \to M_{-1}(X) \to 0$$

We let $\mathcal{O}(X)^\times$ act on the middle term by translations, through $(u, m) \mapsto U^*(m)$ where $U : \mathbb{G}_m \times X \cong \mathbb{G}_m \times X$ is the automorphism multiplication by the unit $u \in \mathcal{O}(X)^\times$. The left inclusion is equivariant if we let $\mathcal{O}(X)^\times$ act trivially on $M(X)$. Thus M_{-1} gets in this way a canonical and functorial structure of $\mathbb{G}_m$-module.

Lemma 3.49. *If M is strongly $\mathbb{A}^1$-invariant, the canonical structure of $\mathbb{G}_m$-modules on M_{-1} is induced from a $\underline{\mathbf{K}}_0^{MW}$-module structure on M_{-1} through the morphism of sheaves (of sets) $\mathbb{G}_m \to \underline{\mathbf{K}}_0^{MW}$ which maps a unit u to its symbol $< u >= \eta[u]+1$. Moreover the pairing of Lemma 3.48, for $n \geq 2$*

$$\underline{\mathbf{K}}_n^{MW} \times M_{-n+1} \to M_{-1}$$

is $\underline{\mathbf{K}}_0^{MW}$-bilinear: for units u, v and an element $m \in M_{-2}(F)$ one has:

$$< u > ([v].m) = (< u > [v]).m = [v].(< u > .m)$$

Proof. The sheaf $X \mapsto M(\mathbb{G}_m \times X)$ is the internal function object $M^{\mathbb{Z}(\mathbb{G}_m)}$ in the following sense: it has the property that for any sheaf of abelian groups N one has a natural isomorphism of the form

$$Hom_{\mathcal{A}b_k}(N \otimes \mathbb{Z}(\mathbb{G}_m), M) \cong Hom_{\mathcal{A}b_k}(N, M^{\mathbb{Z}(\mathbb{G}_m)})$$

where $\mathcal{A}b_k$ is the abelian category of sheaves of abelian groups on Sm_k and $\otimes$ is the tensor product of sheaves of abelian groups. The above exact sequence corresponds to the adjoint of the split short exact sequence

$$0 \to \tilde{\mathbb{Z}}(\mathbb{G}_m) \to \mathbb{Z}(\mathbb{G}_m) \to \mathbb{Z} \to 0$$

This short exact sequence is an exact sequence of $\mathbb{Z}(\mathbb{G}_m)$-modules (but non split as such !) and this structure induces exactly the structure of $\mathbb{Z}(\mathbb{G}_m)$-module on $M^{\mathbb{G}_m}$ and M_{-1} that we used above.

In other words, the functional object $M^{\tilde{Z}(\mathbb{G}_m)}$ is isomorphic to M_{-1} as a $\mathbb{Z}(\mathbb{G}_m)$-module, where the structure of $\mathbb{Z}(\mathbb{G}_m)$-module on the sheaf $\tilde{Z}(\mathbb{G}_m)$ is induced by the tautological one on $\mathbb{Z}(\mathbb{G}_m)$.

Now as M is strongly $\mathbb{A}^1$-invariant the canonical morphism

$$M^{\underline{\mathbf{K}}_1^{MW}} \to M_{-1} = M^{\tilde{Z}(\mathbb{G}_m)}$$

induced by $\tilde{Z}(\mathbb{G}_m) \to \underline{\mathbf{K}}_1^{MW}$, is an isomorphism. Indeed given any N a morphism $N \otimes \tilde{\mathbb{Z}}(\mathbb{G}_m) \to M$ factorizes uniquely through $N \otimes \mathbb{Z}(\mathbb{G}_m) \to N \otimes \underline{\mathbf{K}}_1^{MW}$ as the morphism $\tilde{\mathbb{Z}}(\mathbb{G}_m) \to \underline{\mathbf{K}}_1^{MW}$ is the universal one to a strongly $\mathbb{A}^1$-invariant sheaf by Theorem 3.37.

Now the morphism $\tilde{\mathbb{Z}}(\mathbb{G}_m) \to \underline{\mathbf{K}}_1^{MW}$ is $\mathbb{G}_m$-equivariant where $\mathbb{G}_m$ acts on $\underline{\mathbf{K}}_1^{MW}$ through the formula on symbols $(u,[x]) \mapsto [ux]-[u]$. Now this action factors through the canonical action of $\underline{\mathbf{K}}_0^{MW}$ by the results of Sect. 3.1 as in $\underline{\mathbf{K}}_1^{MW}$ one has $[ux]-[u] =< u > [x]$.

The last statement is straightforward to check. □

For $n \geq 2$ we thus get also on M_{-n} a structure of $\underline{\mathbf{K}}_0^{MW}$-module by expressing M_{-n} as $(M_{-n+1})_{-1}$. However there are several ways to express it this way, one for each index in $i \in \{1, \dots, n\}$, by expressing $M_{-n}(X)$ as a quotient of $M((\mathbb{G}_m)^n \times X)$ and letting $\mathbb{G}_m$ acts on the given i-th factor. One shows using the results from Sect. 3.1 that this action doesn't depend on the factor one chooses. Indeed given $F_0 \in \mathcal{F}_k$ and $u \in M_{-n}(F_0)$, we may see u as a morphism of pointed sheaves (over F_0) $u : (\mathbb{G}_m)^{\wedge n} \to M|_{F_0}$ and Theorem 3.37 tells us that u induces a unique $u' : \underline{\mathbf{K}}_n^{MW} \to M|_{F_0}$. Now the action of a unit $\alpha \in (F_0)^\times$ on u through the i-th factor of $M((\mathbb{G}_m)^n \times X)$ corresponds to letting α acts through the i-th factor $\mathbb{Z}(\mathbb{G}_m)$ of $(\mathbb{Z}(\mathbb{G}_m))^{\otimes n}$ and compose with $(\mathbb{Z}(\mathbb{G}_m))^{\otimes n} \to \underline{\mathbf{K}}_n^{MW} \to M$. A moment of reflexion shows that this action of α on a symbol $[a_1, \dots, a_n] \in K_n^{MW}(F_0)$ is explicitly given by $[a_1, \dots, \alpha.a_i, \dots, a_n] - [a_1, \dots, \alpha, \dots, a_n] \in K_n^{MW}(F_0)$. Now the formulas in Milnor–Witt K-theory from Sect. 3.1 show that this is equal to

$$[a_1] \dots (< \alpha > .[a_i]) \dots [a_n] =< \alpha > [a_1, \dots, a_n]$$

which doesn't depend on i.

This structure of $\underline{\mathbf{K}}_0^{MW} = \underline{\mathbf{GW}}$-module on sheaves of the form M_{-1} will play an important role in the next sections. We may emphasize it with the following observation. Let F be in $\mathcal{F}_k$ and let v be a discrete valuation on F, with valuation ring $\mathcal{O}_v \subset F$. For any strongly $\mathbb{A}^1$-invariant sheaf of abelian groups M, each non-zero element μ in $\mathcal{M}_v/(\mathcal{M}_v)^2$ determines by Corollary 2.35 a canonical isomorphism of abelian groups

$$\theta_\mu : M_{-1}(\kappa(v)) \cong H^1_v(\mathcal{O}_v; M)$$

Lemma 3.50. *We keep the previous notations. Let $\mu' = u.\mu$ be another non zero element of $\mathcal{M}_y/(\mathcal{M}_y^2)$ and thus $u \in \kappa(y)^\times$. Then the following diagram is commutative:*

$$\begin{array}{ccc} M_{-1}(\kappa(v)) & \overset{<u>}{\cong} & M_{-1}(\kappa(v)) \\ \theta_\mu \ \downarrow & & \theta_{\mu'} \ \downarrow \\ H^1_v(\mathcal{O}_v; M) & = & H^1_v(\mathcal{O}_v; M) \end{array}$$

The proof is straightforward and we leave the details to the reader.