MILNOR-WITT K-THEORY

KYLE ORMSBY

Recall that Milnor K-theory of a field F is the quotient of the free associative algebra on $[F^\times] = \{[a] \mid a \in F^\times\}$ which imposes the

logarithm relation: $[ab] = [a] + [b]$ for $a, b \in F^\times$, and
Steinberg relation: $[a][b] = 0$ for $a, b \in F^\times$ such that $a + b = 1$.

Milnor K-theory inherits a grading from word-length in $[F^\times]$ so that the degree of $[a]$ is 1. The resulting graded commutative ring is denoted $K_*^M(F)$.

Similarly, Milnor-Witt K-theory of F is defined as a quotient of the free associative algebra on $[F^\times] \amalg \{\eta\}$ in which $\deg[a] := 1$ and $\deg \eta := -1$. In this case, we impose the

η-twisted logarithm relation: $[ab] = [a] + [b] + [a][b]\eta$ for $a, b \in F^\times$,
Steinberg relation: $[a][b] = 0$ for $a, b \in F^\times$ such that $a + b = 1$,
commutativity relation: $\eta[a] = [a]\eta$ for $a \in F^\times$, and
Witt relation: $(2 + [-1]\eta)\eta = 0$.

The relations are all homogeneous, so the resulting ring $K_*^{MW}(F)$ is $\mathbb{Z}$-graded. (It retains a form of graded commutativity that we will comment upon shortly.)

If we kill η in the presentation of $K_*^{MW}(F)$, we see that we recover Milnor K-theory.

Proposition 1. *For any field F,*

$$K_*^{MW}(F)/(\eta) \cong K_*^M(F).$$

A number of other elementary properties of $K_*^{MW}(F)$ will become clear if we set $\langle a\rangle := 1 + [a]\eta$, $h = 1 + \langle -1\rangle$, and $\varepsilon := -\langle -1\rangle$. Since it plays a distinguished role in the theory, let us also set $\rho := [-1]$ and note that $\varepsilon = -(1 + \rho\eta)$ and the Witt relation can be rewritten as $(2 + \rho\eta)\eta = 0$. Since $\eta - \eta\varepsilon = (1 + (1 + \rho\eta))\eta$, we see that the Witt relation is equivalent to $\varepsilon\eta = \eta$. Since $h = 2 + \rho\eta$, we can also realize the Witt relation as $h\eta = 0$.

The choice of notation is not a feint: $\langle a\rangle \in K_0^{MW}(F)$ indeed corresponds to the quadratic form aX^2, in which case h corresponds to the hyperbolic plane, and the terminology "Witt relation" resolves itself.[1] Following the treatment in [**?**, Chapter 3], we will build up to an isomorphism $K_0^{MW}(F) \cong GW(F)$ in a series of lemmas.

Lemma 2. *For all $a, b \in F^\times$,*

(i) $[ab] = [a] + \langle a\rangle[b] = [a]\langle b\rangle + [b]$,
(ii) $\langle ab\rangle = \langle a\rangle\langle b\rangle$,
(iii) $\langle 1\rangle = 1 \in K_0^{MW}(F)$ *and* $[1] = 0 \in K_1^{MW}(F)$, *and*
(iv) $[a/b] = [a] - \langle a/b\rangle[b]$.

Proof. For (i), simply compute

$$[a] + \langle a\rangle[b] = [a] + (1 + [a]\eta)[b] = [a] + [b] + [a][b]\eta = [ab]$$

Date: July 10, 2015.

[1] The Witt relation then says that h kills η. In the Witt ring $W(F)$, the image of the hyperbolic plane is 0, foreshadowing some relationship between $K_*^{MW}(F)$ in negative grading and $W(F)$.

where the final equality is the η-twisted logarithm relation.

For (ii), multiply the η-twisted logarithm relation by η to get

$$[ab]\eta = [a]\eta + [b]\eta + [a][b]\eta^2.$$

Adding 1 to both sides and liberally applying the commutativity relation, we see that

$$\langle ab\rangle = 1 + [a]\eta + [b]\eta + [a][b]\eta^2 = (1 + [a]\eta)(1 + [b]\eta) = \langle a\rangle\langle b\rangle.$$

For (iii), observe that $\langle 1\rangle - 1 = [1]\eta$, and by the Witt relation, $[1]\eta h = 0$, *i.e.*,

$$(\langle 1\rangle - 1)(\langle -1\rangle + 1) = 0.$$

By (ii), we can expand the left-hand side to get

$$\langle -1\rangle + \langle 1\rangle - \langle -1\rangle - 1 = 0,$$

implying that $\langle 1\rangle = 1$. Now by (i), $[1] = [1] + \langle 1\rangle[1] = [1] + [1]$, and subtracting $[1]$ from both sides we see $[1] = 0$.

Property (iv) is now an easy consequence of the others. □

Lemma 3. *For all $a, b \in F^\times$,*

(i) $[a][-a] = 0$ *and* $\langle a\rangle + \langle -a\rangle = h$*,*
(ii) $[a][a] = [a][-1] = \varepsilon[a][-1] = [-1][a] = \varepsilon[-1][a]$*,*
(iii) $[a][b] = \varepsilon[b][a]$*, and*
(iv) $\langle a^2\rangle = 1$*.*

Proof. To prove the first part of (i), note that $-a = (1-a)/(1-1/a)$, whence $[-a] = [1-a] - \langle -a\rangle[1-1/a]$ by Lemma 2(i). Multiplying on the left by $[a]$, we get

$$\begin{aligned}[-a] &= [a][1-a] - \langle -a\rangle[a][1-1/a]\\ &= 0 - \langle -a\rangle[a][1-1/a]\\ &= \langle -a\rangle\langle a\rangle[1/a][1-1/a]\\ &= 0.\end{aligned}$$

We now prove (ii) before returning to the second part of (i). Since $[-a] = [-1] + \langle -1\rangle[a]$, we can multiply by $[a]$ on the left (and recall that $[a][-a] = 0$) to get

$$0 = [a][-1] + \langle -1\rangle[a][a],$$

in which case

$$[a][a] = -\langle -1\rangle[a][-1] = \varepsilon[a][-1].$$

Note, though, that $0 = [1] = [-1] + \langle -1\rangle[-1]$, so we also know that $-\langle -1\rangle[-1] = [-1]$, in which case

$$[a][a] = [a][-1]$$

as well.

To get the other two equalities in (ii), again start with $[-a] = [-1] + \langle -1\rangle[a]$ and multiply by $[a]$ on the right. The argument is similar.

We now return to the second claim from (i) and compute

$$\begin{aligned}\langle a\rangle + \langle -a\rangle &= 1 + [a]\eta + 1 + [-a]\eta \\ &= 2 + ([a] + [-a])\eta \\ &= 2 + ([-a^2] - \eta[a][-a])\eta \\ &= 2 + [-a^2]\eta \\ &= 2 + ([-1] + [a^2] + [-1][a^2]\eta)\eta \\ &= h + [a^2](1 + [-1]\eta)\eta \\ &= h - [a^2]\eta \\ &= h - (2[a] + [a][a]\eta)\eta \\ &= h - (2[a] + [a][-1]\eta)\eta \\ &= h - [a](2 + [-1]\eta)\eta \\ &= h.\end{aligned}$$

To prove (iii), start with $0 = [ab][-ab]$ and then expand according to Lemma 2(i) to get

$$0 = ([a] + \langle a\rangle[b])([-a] + \langle -a\rangle[b]).$$

Expanding this (and using centrality of $K_0^{MW}(F)$, (i), and multiplicativity of $\langle\rangle$), we get

$$0 = \langle a\rangle([b][-a] + \langle -1\rangle[a][b]) + \langle -1\rangle[-1][b].$$

Replacing $[-a]$ with $[a] + \langle a\rangle[-1]$, we get

$$0 = \langle a\rangle([b][a] + \langle -1\rangle[a][b]) + ([b][-1] + \langle -1\rangle[-1][b]).$$

By (ii), we know that the last term is 0. Multiplying by $\langle 1/a\rangle$ and using $\langle 1\rangle = 1$, we finally get that

$$0 = [b][a] - \varepsilon[a][b],$$

giving the result.

To prove (iv), first observe that $[a^2] = 2[a] + [a][a]\eta = 2[a] + [-1][a]\eta$. Factoring out an $[a]$, we get

$$[a^2] = (2 + [-1]\eta)[a] = h[a].$$

Thus

$$\langle a^2\rangle = 1 + [a^2]\eta = 1$$

by the Witt relation. □

We are now in a good position to elucidate the relationship between $K_0^{MW}(F)$ and the Grothendieck-Witt ring, $GW(F)$. First recall that $GW(F)$ is isomorphic to the quotient of the free commutative algebra on $\{\langle a\rangle \mid a \in F^\times\}$ by the ideal encoding the relations

$$\begin{aligned}\langle ab^2\rangle &= \langle a\rangle, \\ \langle a\rangle + \langle -a\rangle &= \langle 1\rangle + \langle -1\rangle, \text{ and} \\ \langle a\rangle + \langle b\rangle &= \langle a+b\rangle + \langle (a+b)ab\rangle\end{aligned} \tag{4}$$

for any $a, b \in F^\times$ (and $a + b \neq 0$ for the final relation).

Let $W(F) = GW(F)/(h) = GW(F)/\mathbb{Z}h$ denote the Witt ring, and let $\dim : GW(F) \to \mathbb{Z}$ and $\dim_0 : W(F) \to \mathbb{Z}/2\mathbb{Z}$ denote the dimension and mod 2 dimension homomorphisms, respectively. The fundamental ideals in the Grothendieck-Witt and Witt rings are $GI(F) = \ker \dim$ and $I(F) = \ker \dim_0$, respectively. We have

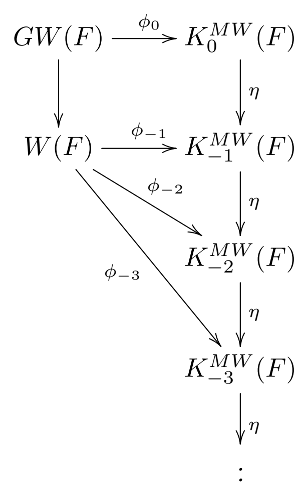
$$\begin{CD} 0 @>>> GI(F) @>>> GW(F) @>{\dim}>> \mathbb{Z} @>>> 0 \\ @. @V{\cong}VV @VVV @VVV @. \\ 0 @>>> I(F) @>>> W(F) @>>{\dim_0}> \mathbb{Z}/2\mathbb{Z} @>>> 0 \end{CD}$$

so that

$$GW(F) \cong W(F) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}$$

is a pullback ring.

Proposition 5. *The assignment $a \mapsto \langle a \rangle$ induces isomorphisms ϕ_n, $n \le 0$, which fit into the commutative diagram*



in which the left-hand vertical map is reduction mod (h).

Proof of well-definition. We first check that ϕ_0 is well-defined. The well-definition of ϕ_n immediately follows since η^{-n} kills h. Lemma 2(ii) and Lemma 3(iv) imply that $\langle ab^2 \rangle = \langle a \rangle \in K_0^{MW}(F)$. Lemma 3(i) says that $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle \in K_0^{MW}(F)$. As for the final relation in Equation 4, first note that we may assume $a + b = 1$. Indeed, otherwise we may multiply by $\langle 1/(a+b) \rangle$, effectively replacing a with $a/(a+b)$ and b with $b/(a+b)$. Now if $a + b = 1$, we only need to check that

$$\langle a \rangle + \langle b \rangle = \langle 1 \rangle + \langle ab \rangle,$$

i.e. that

$$2 + ([a] + [b])\eta = 2 + [ab]\eta.$$

But $[a] + [b] = [ab] - [a][b]\eta = [ab]$ by the Steinberg relation, so the equation holds and ϕ_0 is well-defined. ∎

We now pause the proof of Proposition 5 to introduce Morel's J^*-construction which will play a prominent role in our study of Milnor-Witt K-theory (and which we will use in order to finish our proof). First recall Milnor's homomorphisms

$$\alpha_n : K_n^M(F) \to I^n(F)/I^{n+1}(F)$$

which take symbols $[a_1, \ldots, a_n]$ to $\langle\langle -a_1, \ldots, -a_n \rangle\rangle + I^{n+1}(F)$ where $\langle\langle b \rangle\rangle = \langle 1, b \rangle$ and $\langle\langle b_1, \ldots, b_n \rangle\rangle = \langle\langle b_1, \ldots, b_{n-1} \rangle\rangle \otimes \langle\langle b_n \rangle\rangle$. (These are so-called *Pfister forms.*) For notational convenience, let us write $i^n(F) \coloneqq I^n(F)/I^{n+1}(F)$.[2]

We may now form the pullback

$$\begin{array}{ccc} J^n(F) & \longrightarrow & K_n^M(F) \\ \downarrow & & \downarrow \\ I^n(F) & \longrightarrow & i^n(F) \end{array}$$

i.e.,

$$J^n(F) \coloneqq I^n(F) \times_{i^n(F)} K_n^M(F).$$

If we interpret $I^n(F)$ as $W(F)$ for $n \le 0$, then we get $J^n(F) = W(F)$ for $n < 0$, and

$$J^0(F) = W(F) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z} \cong GW(F).$$

Continuing to drastically overload symbols in the name of future notational simplicity, let us define $\eta \coloneqq 1 \in J^{-1}(F) = W(F)$. Further, let $[a] \coloneqq (\langle a \rangle - 1, [a]) \in J^1(F) \subset I(F) \times K_1^M(F)$. It is straightforward to check that the η-twisted logarithm, Steinberg, commutativity, and Witt relations hold amongst these terms in $J^*(F)$. Thus we get a homomorphism of graded rings $K_*^{MW}(F) \to J^*(F)$ taking η to η and $[a]$ to $[a]$. It is clear that this map is surjective in degrees $n \le 0$, and the reader may check that (by germanely adding copies of the hyperbolic plane), it is surjective in degree $n = 1$. It follows that $K_*^{MW}(F) \to J^*(F)$ is surjective in all degrees.

Proof of Proposition 5, continued. We have already noted that ϕ_n is surjective for $n \le 0$. Let $n < 0$ and note that the composite $W(F) = J^n(F) \to K_n^{MW}(F) \xrightarrow{\phi_n} W(F)$ is the identity. It follows that ϕ_n is injective as well. In case $n = 0$, we again have that $GW(F) \cong J^0(F) \to K_n^{MW}(F) \xrightarrow{\phi_0} GW(F)$ is the identity, whence ϕ_0 is an isomorphism. We conclude that ϕ_n is an isomorphism for all $n \le 0$. □

In fact, more is true.

Theorem 6. *The canonical map $K_*^{MW}(F) \to J^*(F)$ is an isomorphism.*

*Proof (following [**?**]).* □

[2] Of course, by the affirmative resolution of Milnor's conjecture on quadratic forms, $i^n(F) \cong k_n^M(F)$ where $k_n^M(F) \coloneqq K_n^M(F)/(2)$, but we will not use this for some time.