Merling - Equivariant algebraic K-theory

Note Title 5/30/2015

Rings & G, Schemes, Spaces $\xrightarrow{K_i}$ Ab

Rings & G, Schemes, Spaces $\xrightarrow{A}$ connective Ω-spectra $\xrightarrow{\pi_i}$ Ab

We want $K_G R \simeq \Omega^V X_V$ for G-spectrum X

If G acts on a ring R, it acts on R-modules via scalars.

Given a map $R \longrightarrow S$ of G-action leads to

$\mathrm{Mod}(R) \xrightarrow{\otimes_R S} \mathrm{Mod}(S)$ not a G-module.

$(gM)\otimes_R S \neq g(M\otimes_R S)$ but they are isomorphic.

Recall KR = gp completion of $B(\text{iso}\, P(R))$ ← fin gen proj R-modules

$P(R)$ is a symm monoidal category.

It leads to Ω-spectra via May operad machine or Segal's Γ space machine

Then agree by May-Thomason theorem.

Equivariant analogs:

operad machine : Guillou-May

Γ spaces : Shimakawa 1980s

May-Thomason thm analog proved by
May-Merling-Osorno.

The input is a "genuine" symm monoid category with G-action, which I will explain.

Assume G is finite. What does "genuine" mean?

transfers? $C^H \underset{tr}{\overset{res}{\rightleftarrows}} C^G$

$$\prod_{G/H} C \longrightarrow C$$

$j = |G/H|$ $\quad c_1, \dots, c_j \longmapsto c_1 \oplus c_2 \cdots \oplus c_j$

not equivariant since G permutes on the left but not on the right.

Intuition Recall 0th component of KR is

$$\coprod_n BGL_n(R)$$

Idea: replace these by equiv bundles.

Thm (Guillou–May–M) G finite, Π compact Lie

$B\mathrm{Cat}(\tilde{G}, \tilde{\Pi}) \to B\mathrm{Cat}(\tilde{G}, \Pi)$ is a univ princ $\Pi \rtimes G$-bundle

$\tilde{G}$ = translation category of G

$B\tilde{G} \simeq EG$

$\mathrm{Cat}(P, D)$ = all functors + nat trans of G-categories

Guess $K_G(R)$ = gp completion of $\coprod B\mathrm{Cat}(\tilde{G}, GL_n(R))$

Is this an Ω^∞ G-space?

Def $\mathrm{Cat}(\tilde{G}, \mathcal{C})^{G} := \mathcal{C}^{hG}$
= hty fixed points of $\mathcal{C}$

E_∞-operads

	Top	Cat
non equiv	$\mathcal{O}(j) = E\Sigma_j$	$B\mathcal{O}(j) = E\Sigma_j$ $\mathcal{O}(j) = \tilde{\Sigma}_j$
equiv	$\mathcal{O}_G$ = universal $G\text{-}\Sigma_j$-bundle	$\mathcal{O}_G(j) = \mathrm{Cat}(\tilde{G}, \tilde{\Sigma}_j)$ (Guillou–May)

FACT (May): permutative cats $\simeq$ alg / $\mathcal{O}$ ← strict SMC

symm monoidal cats $\simeq$ pseudo alg / $\mathcal{O}$.

Def (Guillou–May) A genuine perm G-cat is an algebra / $\mathcal{O}_G$

A genuine SMC is a pseudo-algebra / $\mathcal{O}_G$.