$RO(G)$ - graded "ordinary" $\mathbb{Z}/2$ - cohomology

Note Title 5/30/2015

for $G = (\mathbb{Z}/2)^n$

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$\underbrace{RO(G)}_{\text{real representation ring of } G,\ G \text{ finite}}$ - graded equivariant spectrum : (generalised cohomology theory)

$$E^V(\underset{G\text{-space}}{X}) = [X, E_V]_G$$

$V \in RO(G)$

$V + W = V' \in RO(G) \qquad E_V \simeq \Omega^W E_{V'} =$

$= F_{\text{based}}(S^W, E_{V'})$.

May rigidification: $\mathcal{U} = \bigoplus_{\infty}$ all irreducible G-reps.

E_V $\quad V \subset \mathcal{U}$ finite G-representation

$V + W = V'$

orthogonal sum $\quad \sigma_V^{V'} : E_V \xrightarrow{\cong} \Omega^W E_{V'}$ $\quad \cdots\cdots$

<u>Fixed points of a G-spectrum</u> E

$$(E^G)_n := (E_n)^G$$

$\{e\}$-spectrum = spectrum $\quad$ trivial n-dim real G-rep.

"Geometric" fixed points

Lewis, May + Steinberger
LNM 1213

Greenlees
(Adams ?)

$$\Phi^G E = \varinjlim_{\substack{V \subset \mathcal{U} \\ \text{finite.}}} \Omega^{V^G} (E_V)^G$$

" take fixed point of all E_V's
but also, when we consider
$V + W = V'$
$\rightsquigarrow V^G + W^G = (V')^G$."

Example: $\Phi^G S_G = S$.

G-suspension spectrum of a sphere

It turns out, Φ^G commutes with spectrification.

$\therefore$ We can compute $\Phi^G E$ by using a (inclusion) pre-spectrum.

tom Dieck $\to$ computing Φ^G of all kinds of equivariant cobordism.

$\Phi^G E$ can help with computing E^G.

A family $\mathcal{F}$ of subgroups of G : a set of subgroups closed under sub conjugation.

$E\mathcal{F}$ G-CW-complex,

$$E\mathcal{F}^H \simeq * \quad \text{if } H \in \mathcal{F}$$
$$\emptyset \quad \text{else}$$

Ex: $\mathcal{F}[H] = \{$all subgroups of G not containing $H\}$.

$H \triangleleft G$

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}}$$

mapping cone.

Observation:

E spectrum:

$$\Phi^G E = \big(\underbrace{\widetilde{E\mathcal{F}[G]} \wedge E}_{\text{spectrum}}\big)^G. \quad \square$$

If $G = \mathbb{Z}/p$, $\mathcal{F}[\mathbb{Z}/p] = \{\{e\}\}$

$$E\mathbb{Z}/p_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}[\mathbb{Z}/p]}$$

$$\underbrace{(E\mathbb{Z}/p_+ \wedge E)^{\mathbb{Z}/p}} \longrightarrow E^{\mathbb{Z}/p} \longrightarrow \Phi^{\mathbb{Z}/p} E$$

cofibration sequences

Adams isomorphism: $\cong$ $E\mathbb{Z}/p_+ \wedge_{\mathbb{Z}/p} E$ Borel E-homology

"use E_∞ only"

E ring spectrum

split. $E^{\mathbb{Z}/p} \longrightarrow E_{\{e\}}$

right inverse

$E\mathbb{Z}/p_+ \wedge_{\mathbb{Z}/p} E \simeq E_{\{e\}} \wedge B\mathbb{Z}/p_+$

This solves $MU_{\mathbb{Z}/p}$, if you can get the connecting map

$$\underbrace{\Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p}}_{\text{tom Dieck}} \longrightarrow \Sigma B\mathbb{Z}/p_+ \wedge MU.$$

"the Tate diagram"

$$\begin{tikzcd}
E\mathbb{Z}/p_+ \wedge_{\mathbb{Z}/p} E \arrow[r] \arrow[d, "\sim"'] & E^{\mathbb{Z}/p} \arrow[r] \arrow[d] & \Phi^{\mathbb{Z}/p} E \arrow[d] \\
E\mathbb{Z}/p_+ \wedge_{\mathbb{Z}/p} F(E\mathbb{Z}/p_+, E) \arrow[r, "N"'] & F(E\mathbb{Z}/p_+, E)^{\mathbb{Z}/p} \arrow[r] & \hat{E}
\end{tikzcd}$$

Tate square,

Tate cohomology

Spectral sequences: Greenlees & May: Generalised Tate cohomology

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If G is abelian finite, the Tate square has a generalization: The isotropy separation

Spectral sequence:

ISSS: $\mathcal{P}$ poset of non-empty sets

$$T = \{H_1, \dots H_k\}, \quad \{e\} \subseteq H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_k \subseteq G$$

with respect to inclusion

$$\Gamma : T \mapsto$$

$$F(EG/H_{1+}, \dots \quad E\mathcal{F}[\widetilde{H_{k-1}}] \wedge F(EG/H_k, E\mathcal{F}[\widetilde{H_k}] \wedge E)\dots)$$

Exercise: For $G = \mathbb{Z}/p$, you get the data for the Tate square:

$$\begin{tikzcd} & \Phi^{\mathbb{Z}/p} E \arrow[d] \\ F(E\mathbb{Z}/p_+, E) \arrow[r] & E \end{tikzcd}$$

<u>Theorem</u> 1 (Abram, K.): $E \longrightarrow \underleftarrow{\mathrm{holim}}\ \Gamma(E)$

ISSS

is an equivalence.

<u>Theorem</u> 2: If $E = MU_G$, (G finite abelian) the higher derived functors in the ISSS vanish.

$$MU_{G*} = \underleftarrow{\lim}\ \pi_* \Gamma' MU_G$$

computable.

The situation is a little different in the case of "ordinary" equivariant cohomology.

If M is a G-Mackey functor: $H \leq G$

$$(HM)^H_i = \begin{cases} 0 & \text{if } i \neq 0 \\ M(H) & \text{if } i = 0. \end{cases} \quad i \in \mathbb{Z}$$

Problem:

Find $RO(G)$-graded coefficient of HM

"constant" Mackey functor : $\mathbb{Z}/2$

restrictions $\cong$.

$G = \mathbb{Z}/2$ (Hu, K.) ~ 1998 $RO(\mathbb{Z}/2) = \mathbb{Z}\{1, \alpha\}$ ← sign rep.

We do not know (a priori) $\Phi^{\mathbb{Z}/2} H\mathbb{Z}/2$!

But we do know that

$$\Phi^{\mathbb{Z}/2} \Sigma^{k\alpha} H\mathbb{Z}/2 = \Phi^{\mathbb{Z}/2} H\mathbb{Z}/2$$

← E_∞-ring spectrum

Geom fixed points are only $\mathbb{Z}$-graded.

LES ⊙ ← Tate diagram

$$\begin{tikzcd} H\mathbb{Z}/2_* B\mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \Phi^{\mathbb{Z}/2} H\mathbb{Z}/2_* \arrow[lll, bend left] \end{tikzcd}$$

$$\Phi^{\mathbb{Z}/2} H\mathbb{Z}/2_* = \mathbb{Z}/2[x] \qquad |x| = 1$$

$$H\mathbb{Z}/2_* B\mathbb{Z}/2[k] \longrightarrow \Sigma^{k\alpha} H\mathbb{Z}/2_* \longrightarrow \mathbb{Z}/2[x]$$

$$\Sigma^{k\alpha} H\mathbb{Z}/2_* = \begin{cases} \mathbb{Z}/2 \text{ in dim } 0, \dots, k, & k \geq 0 \\ \mathbb{Z}/2 \text{ in dims } k+2, \dots, 0, & k \leq -2 \\ 0 & k = -1 \end{cases}$$

gap behavior.

Generalised to $\mathbb{Z}/(2^k)$ by Hill, Hopkins, Ravenel

What about $(\mathbb{Z}/2)^n$?

$$RO((\mathbb{Z}/2)^n) = \text{free abelian group of rank } 2^n$$

Computing: $RO(G)$-graded coeffs of

$$(H\mathbb{Z}/2)_{(\mathbb{Z}/2)^n}$$

$$\alpha_1, \ldots, \alpha_{2^k-1} \neq 0 \in \mathrm{Hom}((\mathbb{Z}/2)^k, \mathbb{Z}/2)$$

$$\left(H\mathbb{Z}/2_G \wedge\right) \bigwedge_{i=1}^{2^n-1} \left(EG/\mathrm{Ker}(\alpha_i)_+ \longrightarrow S^0 \right) \Bigg\}$$ iterated cofiber is $\Phi^G H\mathbb{Z}/2$.

$$E(G/H_1)_+ \wedge \cdots \wedge E(G/H_\ell)_+ = E(G/H_1 \cap \cdots \cap H_\ell)_+.$$

1. Spectral sequence for $(\Phi^G H\mathbb{Z}/2)_*$ - E^1 computable
2. Collapses to E^2, the only d^1's come from $\cong$ in the cube, $\therefore$ E^2 computable

<u>Theorem</u> (Holler, K.): $\left(\Phi^{(\mathbb{Z}/2)^k} H\mathbb{Z}/2\right)_* =$

$$= \mathbb{Z}/2[x_\alpha] \Big/ (x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma, \quad \begin{matrix} \alpha+\beta+\gamma=0 \\ \alpha, \beta, \gamma \neq 0 \end{matrix})$$

$\alpha : (\mathbb{Z}/2)^k \to \mathbb{Z}/2 \qquad \alpha \neq 0$

<u>Poincaré series</u>: $\dfrac{1}{(1-x)^k} \displaystyle\prod_{i=1}^{k} \left(1 + (2^{i-1} - 1)x\right)$

<u>Theorem</u>: For $m_\alpha \geq 0 \quad \forall \alpha : (\mathbb{Z}/2)^k \to \mathbb{Z}/2, \ \alpha \neq 0$

the Poincaré series of $(\Sigma^{\Sigma m_\alpha \alpha} H\mathbb{Z}/2)_\star$ is:

$$\frac{1}{(1-x)^k}\left(\sum_{(\mathbb{Z}/2)^j \cong H \subset G^*} (-1)^j \prod_{i=1}^{k-j}\left(1+(2^{i-1}-1)x\right) x^{j+\sum_{\alpha\in H\setminus\{0\}} m_\alpha}\right)$$

If some $m_\alpha < 0$, we have a chain cx. whose homology is the answer.

<u>Combinatorial identity</u>:

$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} x^j \prod_{i=1}^{k-j} \left(1 + (2^{i-1} - 1)x\right) = (1-x)^k$$

$\begin{bmatrix} k \\ j \end{bmatrix}$ = # of j-dim. $\mathbb{Z}/2$-vector subspaces of $(\mathbb{Z}/2)^k$.

$$= \frac{(2^k - 1) \cdot (2^{k-1} - 1) \cdot \ldots \cdot (2^{k-j+1} - 1)}{(2^j - 1) \cdot \ldots \cdot (2^1 - 1)}$$

$$X \wedge Y = X \times Y / (* \times Y) \cup (X \times *)$$

$* \in X$

$* \in Y$

smash product $\Leftarrow$ F. Adams.