Heller - Endomorphisms of the equivariant motivic sphere

Note Title 5/31/2015

joint with Gepner

G = finite gp, k = field with $|G|$ prime to char k

① equiv motivic-ity considered by Voevodsky and Hu-Kriz-Ormsby. Start with smooth scheme$/k$ with G-action.

simplicial presheaves.

$$G\,Sm/k \longrightarrow sPre(G\,Sm/k)$$

Use "global" model structure and impose relations, i.e. AKB locazn.

① force equiv distinguished square become htpy pushouts

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "b"', "\text{equiv étale}"] \\ U \arrow[r, hook, "\text{invariant open}"'] & X \end{tikzcd}$$

isom. $\qquad b^{-1}(X \setminus U) \xrightarrow{\cong} X \setminus U$

② $\underline{X} \times \mathbb{A}^1 \longrightarrow \underline{X}$ is an equiv $\forall\, X \in G\mathrm{Sm}/k$

$\rightsquigarrow \mathcal{H}_G(k)$

Remark Equiv weak equiv are not detected by fixed points; i.e. $X^H \xrightarrow{\sim} Y^H \quad \forall H < G$

$\not\Rightarrow$ f is motivic eqvr equivalence.

P. Hermann showed that eqvr alg K-theory is not an invariant of fixed pointwise equivs.

Motivic rep sphere for rep V/k is

$$T^V = \mathbb{P}(V \oplus \mathbb{1}) / \mathbb{P}(V)$$

= one pt compactfn of V

$S\mathcal{H}_G(k)$ = stabilization of $\mathcal{H}_G(k)$ at T^{ρ_G}

where $\rho_G = k[G]$ = regular rep

Tensor triangulated category where the unit is sphere spectrum $\mathbb{S}_k$

Question What is $\mathrm{End}_{SH_G(k)}(\mathbb{S}_k)$.

Classically (Segal) this is the Burnside ring $A(G)$

Thm (tom Dieck splitting) Let $Y \in \mathcal{T}_G$. Then $\exists$ iso

$$\bigoplus_{(H)} \pi_n^{WH}(\Sigma^\infty EWH_+ \wedge Y^H) \xrightarrow{\simeq} \pi_n^G(\Sigma^\infty Y)$$

Adams iso: $\pi_n^{WH}(\Sigma^\infty EWH_+ \wedge Y^H) \simeq \pi_n(\Sigma^\infty EWH_+ \wedge_{WH} Y^H)$

Specializing to $n=0$, $Y=S^0$

$$\pi_0^G(S^0) = \bigoplus_{(H)} \pi_0\left(\Sigma^\infty BWH_+\right) = \bigoplus_{(H)} \mathbb{Z} \cong A(G)$$

Proof of tom Dieck thm for $G=C_2$

$$\left(EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G\right) \wedge Y \qquad W\{e\} = G, \quad W(G) = \{e\}$$

$\Longrightarrow$ LES in π_*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n^G(EG_+ \wedge Y) \arrow[r] & \pi_n^G(Y) \arrow[r] & \pi_n^G(\tilde{E}G \wedge Y) \arrow[r] & \cdots \\
& & \pi_n Y^G \arrow[u] \arrow[ur, "\cong"'] & &
\end{tikzcd}$$

Geo fixed pts. Φ^G

$$\Phi^{G}\Sigma^\infty X = \Sigma^\infty X^{G}$$

$$\Phi^{G}(\Sigma^\infty Y) = (\tilde{E}G \wedge \Sigma^\infty Y)^{C_2}$$

Thm Let Y be a based motivic G-space. (G any finite gp)

Then $\exists$ iso

$$\Theta : \bigoplus_{(H)} \pi_n^{WH}(\Sigma^\infty \mathbb{E}WH_+ \wedge Y^H) \xrightarrow{\cong} \pi_n^{G}(\Sigma^\infty Y)$$

Cor $\bigoplus_{(H)} \pi_0(\Sigma^\infty BWH_+) \xrightarrow{\cong} \pi_0^{G}(\mathbb{S}_k)$

Here BWH is geometric classifying space

$\pi_0(BWH_+) \neq \mathbb{Z}$

Θ_H is the composite

$$\begin{tikzcd}
\pi_n^{WH}(\Sigma^\infty EWH_+ \wedge Y^H) \arrow[r, "p^*"] \arrow[d, "\Theta_H"'] & \pi_n^{NH}(\Sigma^\infty EWH_+ \wedge Y^H) \arrow[r, "i_*"] & \pi_*^{NH}(\Sigma^\infty EWH_+ \wedge Y) \arrow[d, "w", "\cong"'] \\
\pi_n^G(\Sigma^\infty Y) & & \pi_n^G(G_+ \wedge_{NH} \Sigma^\infty EWH_+ \wedge Y) \arrow[ll, "q"']
\end{tikzcd}$$

w = motivic Wirthmüller iso

p_* is restriction along $p : NH \xrightarrow{\text{quotient}} WH$

$i : Y^H \hookrightarrow Y$

q is collapse (i.e. $G_+ \wedge_{NH} \Sigma^\infty(EWH_+ \wedge Y)$)

$$= (G_+ \wedge_{NH} \Sigma^\infty EWH_+) \wedge Y \longrightarrow S^0 \wedge Y$$

important case : when Y is concentrated at one conj class

(i.e $\exists H < G$ s.t. $Y^K \simeq *$ if $(K) \neq (H)$)

in this case inverse to p^* is geometric fixed points.

Geometric fixed points in motivic land.

Let $\mathcal{F}$ be a family of subgps closed under subconj.

$\mathcal{F}[N]$ = subgps not containing N

$\mathcal{F}_{triv} = \{\{e\}\}$

etc.

Definition given of motivic geometric fixed points.