Guillou - Eta + the structure of motivic Ext

Note Title 5/31/2015

joint with D. Isaksen

Morel-Voevodsky $\mathbb{A}^1$-htpy category $/k$.

There are realzns

$$\begin{array}{ccccc} Mot_{\mathbb{R}} & \longrightarrow & Mot_{\mathbb{C}} & & \underline{X} \\ \downarrow & & \downarrow & & \downarrow \\ C_2\text{-Top} & \longrightarrow & Top & \longrightarrow & X(\mathbb{C}) \end{array}$$

Two circles

simplicial circle $= S^{1,0}$

geometric circle $= S^{1,1} = \mathbb{A}^1 \setminus \{0\}$

$$S^{p,q} = (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge q} \rightsquigarrow \pi_{p,q}X = [S^{p,q}, X]_{\mathbb{A}^1}$$

motivic stable: invert $S^{2,1} \simeq \mathbb{P}^1$

Motivic Hopf maps:

$\eta_{top} : S^{3,0} \longrightarrow S^{2,0} \quad \in \pi^S_{1,0}$

$\eta_{mot} : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$

$\qquad\quad \| $

$\qquad S^{3,2} \longrightarrow S^{2,1} \quad \in \pi^S_{1,1}$

Classically $\eta^4 = 0$, so $\eta_{top}^4 = 0$, but η_{mot} is not nilpotent. Look at C_2-hty 3

$$\mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{C}P^1$$

fixed pts $\quad \underbrace{\mathbb{R}^2 \setminus \{0\}}_{\simeq S^1} \longrightarrow \underbrace{\mathbb{R}P^1}_{= S^1}$ degree 2 map

Morel has a motivic Adams SS, later studied by Dugger-Isaksen and Hu-Kriz-Ormsby.

$$\mathrm{Ext}^{p,q,u}_{\mathcal{A}_{mot}}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \pi_{q-p,u}(\mathbb{S}^{\wedge}_2)$$

motivic H^*, $H^{*,*}(X; \mathbb{F}_2)$

$$\mathbb{M}_2 = H^{*,*}(\mathrm{Spec}\ \mathbb{C}) \cong \mathbb{F}_2[\tau] \quad \text{with } |\tau| = (0,1)$$

$\mathcal{A}_{mot}$ = motivic Steenrod algebra

Thm (Voevodsky) $\mathcal{A}_{mot}$ = $\mathbb{M}_2$-algebra generated by Sq^i for $i>0$ subject to Adem rels similar to classical ones.

$|Sq^{2n}| = (2n, n)$

$|Sq^{2n+1}| = (2n+1, 1)$

$Sq^1 Sq^1 = 0$

$Sq^1 Sq^2 = Sq^3$

$Sq^2 Sq^2 = \tau Sq^3 Sq^1$

$Sq^2 Sq^{14} Sq^6 Sq^2 Sq^2 = \tau^2 Sq^{15} Sq^7 Sq^3 Sq^1$

Remark $\mathcal{A}_{mot}[\tau^{-1}] \simeq \mathcal{A}_{cl}[\tau^{\pm 1}]$ $\mathcal{A}_{cl}$ = classical Steenrod alg.

Isaksen has 70-stem chart

features:

1) vanishes above line of slope $\frac{1}{5}$

use Adams argument of 1965

2) $\exists$ h_1-towers. h_1 detects η_{mot}

3) h_1-local above slope $1/2$

What is $\mathrm{Ext}_{\mathcal{A}_{mot}}[h_1^{-1}]$?

Thm (G.-Isaksen)

$$\mathrm{Ext}_{\mathcal{A}_{mot}}[h_1^{-1}] = \mathbb{F}_2[h_1^{\pm}][v_1^4, v_2, v_3, \ldots]$$

$$|v_1^4| = (8,4) \qquad v_n = (2^{n+1}-2, 2^n-2)$$

Adams differentials

$d_2(v_3) = h_1 v_2^2$ (Classically $d_2(e_0) = h_1^2 d_0$)

$d_2(v_4) = h_1 v_3^2$ $\quad v_4 = e_0 g h_1^3$

Conjecture $d_2(v_n) = h_1 v_{n-1}^2$ for $n \geq 3$ (2014)

This implies $E_3 = E_\infty = \mathbb{F}_2[h_1^{\pm 1}][v_1^4, v_2]/(v_2^2)$

$\Rightarrow \pi_{**}(S[\eta^{-1}]) \simeq \mathbb{F}_2[\eta^{\pm 1}][\mu, \varepsilon]/\varepsilon^2$

It is equiv to $\alpha_1^{-1}(\mathrm{ANSS}\ E_2) \simeq \mathbb{F}_2[\alpha_1^{\pm 1}]\{$classes on 1-line$\}$

This is a thm of Andrews-Miller.

3) Slope 1/2 line (classical Adams 1966)

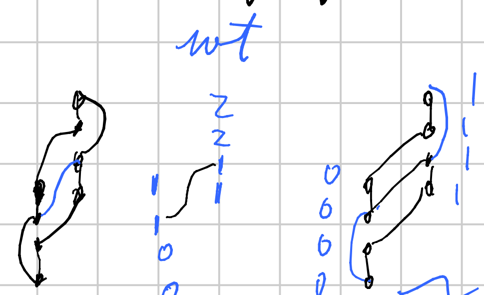
Recall $\mathcal{A}$ has finite subalgebras $\mathcal{A}(0), \mathcal{A}(1), \dots$

$\mathcal{A}(n)$ gen'd by $Sq^1, Sq^2, \dots, Sq^{2^n}$.

Adams reduces problem to $\mathrm{Ext}_{\mathcal{A}}(\mathcal{A}(0), \mathbb{F}_2)$

Shows low degree vanishing for any free $\mathcal{A}(0)$-module.

Let $\tilde{\mathcal{A}} = \mathcal{A} \otimes_{\mathcal{A}(1)} \widetilde{\mathcal{A}(1)}$



$\mathcal{A}(1)$ $\qquad$ $\widetilde{\mathcal{A}(1)}$

Is $\mathcal{A}(1)$ or $\widetilde{\mathcal{A}(1)}$ realizable? They need $\mathcal{A}$-module structures. How does Sq^4 act?

It must connect 1 and $\bar{5}$, and there are 4 possible structures. All are realizable classically. see Davis-Mahowald

Let $Y = C(2) \wedge C(\eta)$. It has v_1-self map $\Sigma^2 Y \xrightarrow{v_1} Y$

There are 4 versions yielding the 4 $\mathcal{A}$-module structures.

Motivic: $Y = C(2) \wedge C(\eta_{mot})$

$\exists$ v_1-self maps $\Sigma^{2,1} Y \xrightarrow{v_1} Y \longrightarrow C(v_1)$

realizing the 4 structures

We also have $Y_1' = C(2) \to C(\eta_{top})$ with

$$\Sigma^{2,1} Y_1' \xrightarrow{\nu_1'} Y \longrightarrow C(\nu_1') \text{ realizing } \widetilde{A(1)}$$

$\eta_{top} = \eta_{mot}$ J.