Turaev-Viro TQFTs via Planar Algebras

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How To Construct a TQFT:

1. Algebraic Input
2. Combinatorial Description of Manifolds
3. Shake them together...

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1. Speherical Tensor Category/Planar Algebra
2. Triangulations of 3-Manifolds
3. The Turaev-Viro Construction...

Theorem (Pachner)

Any two triangulations of d-manifolds can be related by a finite sequence of 'Pachner moves'

$$(k \leftrightarrow d + 2 - k) \qquad\qquad 1 \leq k \leq d/2 + 1$$

Triangulations of Manifolds

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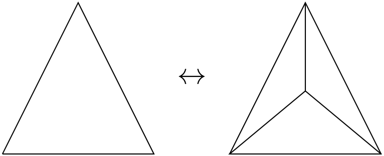
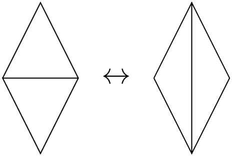


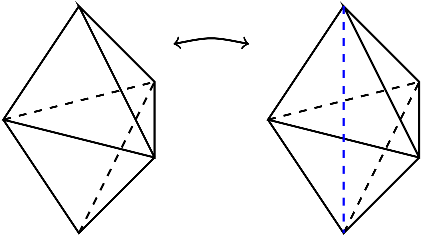
$1 \leftrightarrow 3$



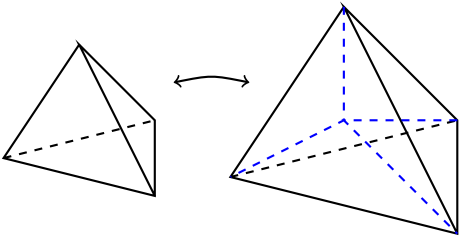
$2 \leftrightarrow 2$

3D Pachner Moves:

The 2-3 Move:



The 1-4 Move:



The Turaev-Viro Construction

Basic Idea: M = 3-manifold, C = Spherical Category.

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- Sum over labelings.
- (Prove invariance under 2-3 moves, 1-4 moves, and reordering)
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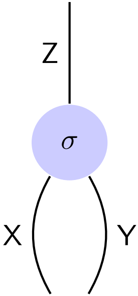
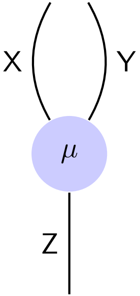
Miracle: This invariant is not trivial!

Properties of Planar Algebras



- Simple Objects $(X, Y, Z, \cdots)$
- Morphisms

Properties of Planar Algebras



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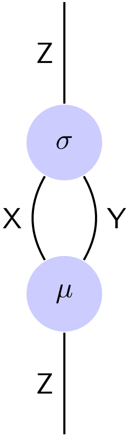

- Simple Objects $(X, Y, Z, \cdots)$
- Morphisms
- Dual Morphisms
- Hermitian Inner Product (positive definite!)
  $\langle \mu, \sigma \rangle$

A Basis $\{\sigma_i\}$ for $Hom(X \otimes Y, Z)$ also gives a daul basis $\{\sigma_i^*\}$! for $Hom(Z, X \otimes Y)$.

Basis for $Hom(X \otimes Y, Z)$ also gives basis for

- $Hom(Y \otimes \overline{Z}, \overline{X})$ (rotate in planar algebra!)

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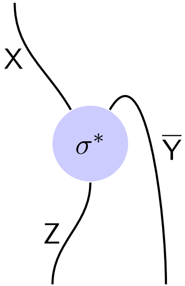
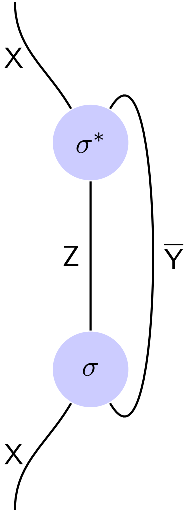
- and $Hom(Z \otimes \overline{Y}, X)$



Is it orthonormal?



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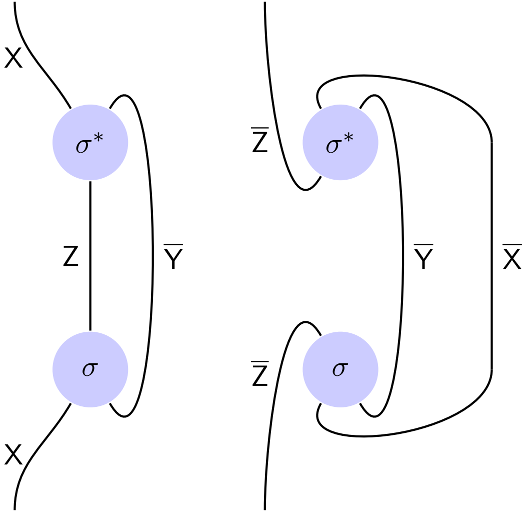
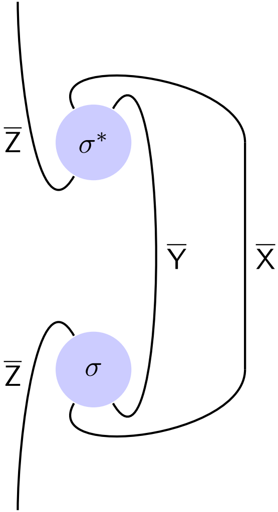


$\cdot d(Z)d(X)^{-1}$



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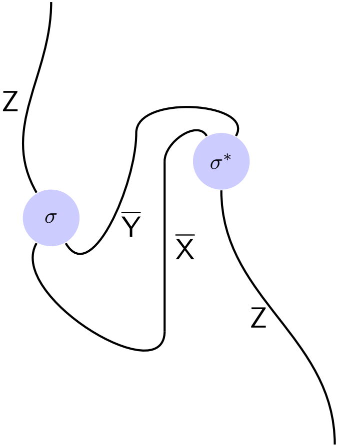

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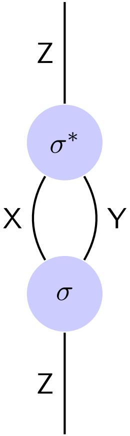
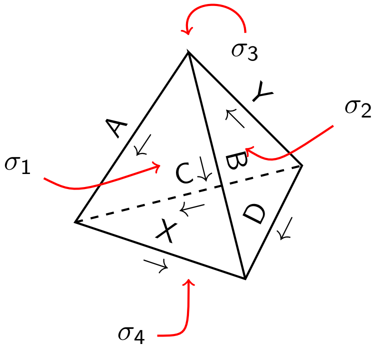


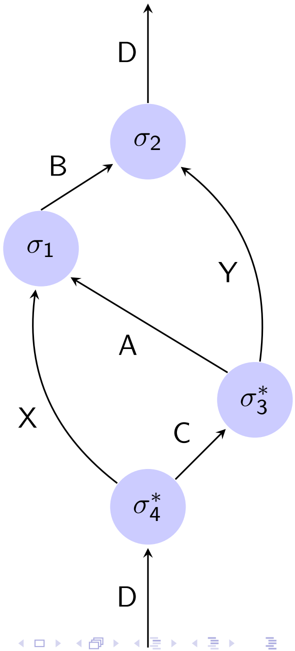
$\cdot d(Z) d(X)^{-1}$

Need to rescale by a factor of $\sqrt{d(Z)^{-1} d(X)}$.

How to Label a Tetrahedron and get a Number

- Choose representative simples $A, B, C, X, Y, Z, \ldots$
- Choose basis $\{\sigma_i\}$ for each $Hom(A \otimes B, C)$.
- Label edges with simple objects.
- Label faces with elements σ_i.
- Then...


σ_3
σ_2
σ_1
σ_4
A
Y
B
C
D
X


D
B
σ_2
σ_1
Y
A
σ_3^*
X
C
σ_4^*
D

Get a formula...

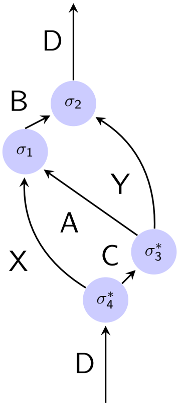
$$Z(M, \underbrace{\mathcal{T}}_{\text{triangulation}}) = \sum_{\substack{\text{edge} \\ \text{lables}}} \sum_{\substack{\text{face} \\ \text{labels}}} \prod_{\substack{\text{tetrahedra} \\ T}} Z(T)$$

If orientation of T *disagrees* with M, use $\overline{Z(T)}$.

How to prove invariance?
Is it even invariant?

Reordering Vertices

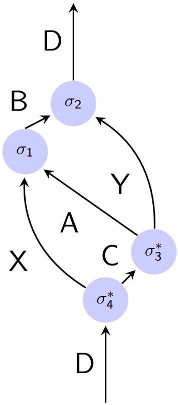
What happens when we reorder the vertices?



Reordering Vertices

What happens when we reorder the vertices?
Original (13)-edge label: B, (02)-edge label: C

- $Z(T_{0\leftrightarrow 1})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(A)d(D)}$
  (13)-edge: D, (02)-edge: A



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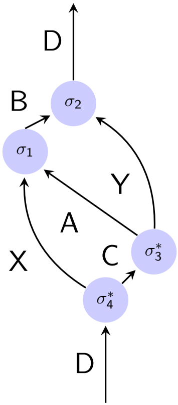
- $Z(T_{1\leftrightarrow 2})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(X)d(Y)}$
  (13)-edge: X, (02)-edge: Y



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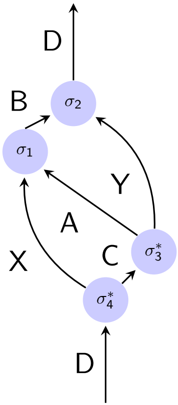
- $Z(T_{1\leftrightarrow 3})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(C)d(B)}$
  (13)-edge: B, (02)-edge: C



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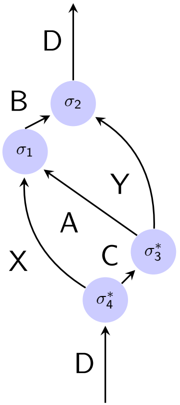
Correction factor: $\sqrt{d(B)^{-1}d(C)^{-1}}$

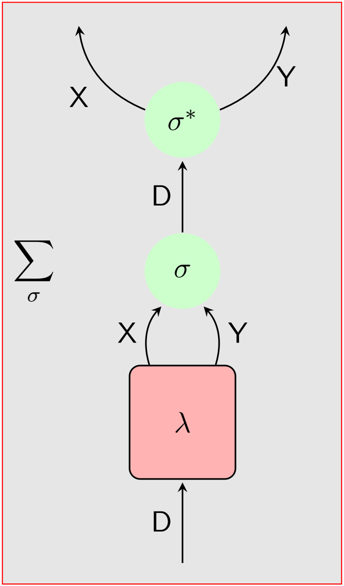


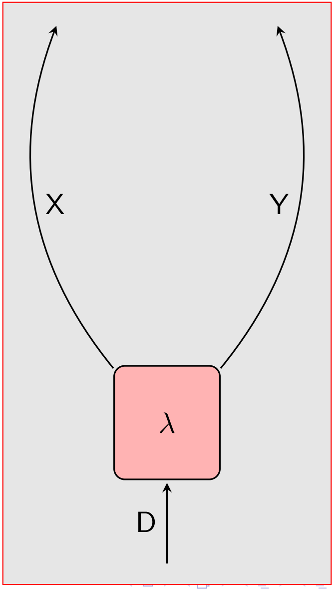
$$Z(M, \underbrace{\mathcal{T}}_{\text{triangulation}}) = \sum_{\substack{\text{edge} \\ \text{lables}}} \sum_{\substack{\text{face} \\ \text{labels}}} \prod_{\substack{\text{tetrahedra} \\ T}} Z_{\text{corrected}}(T)$$

Invariance:

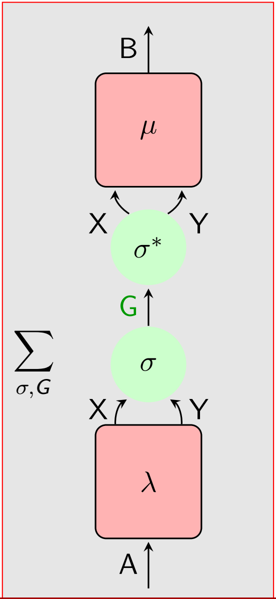
- Vertex order ✓
- 2-3 Pachner Move
- 1-4 Pachner Move

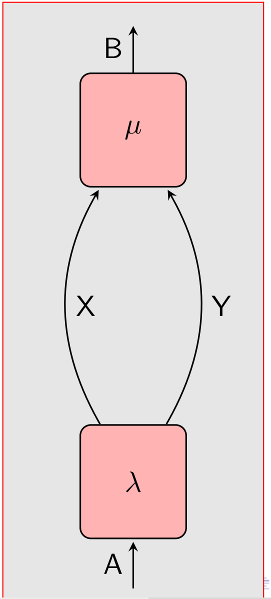
First Identity:



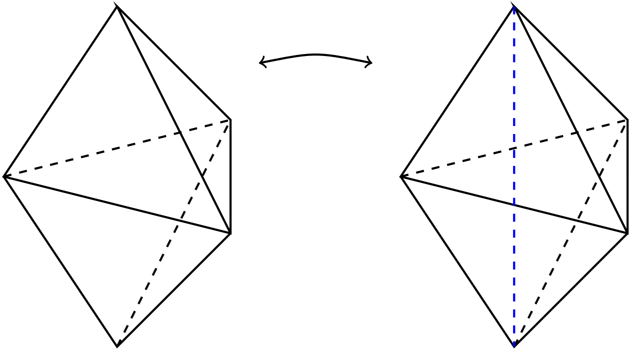


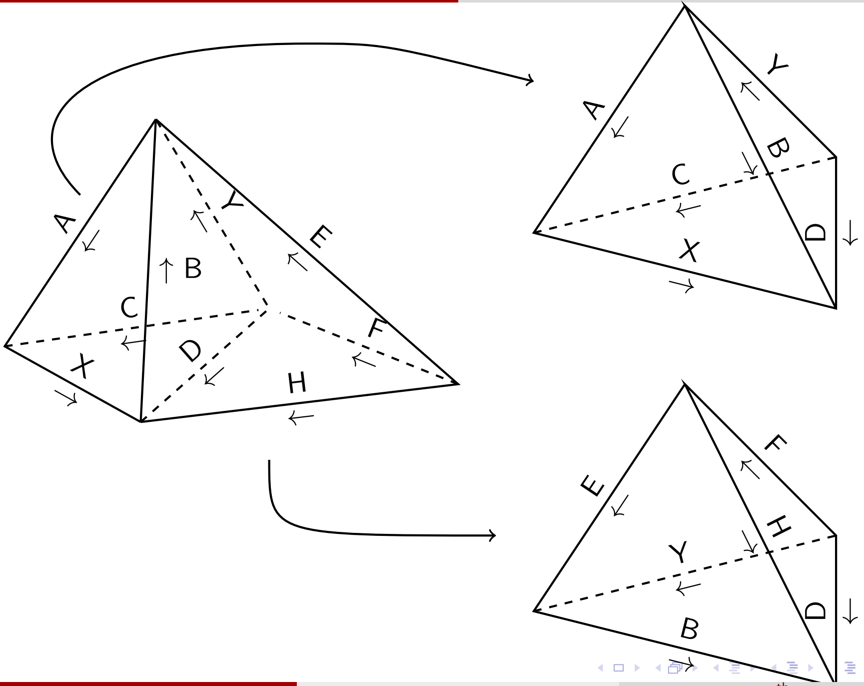
Second Identity:

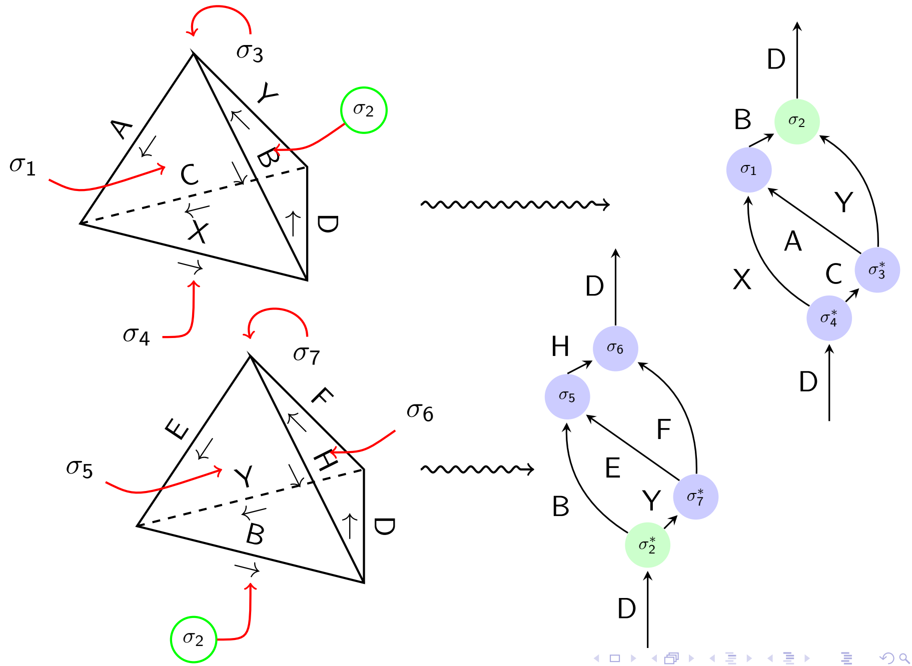


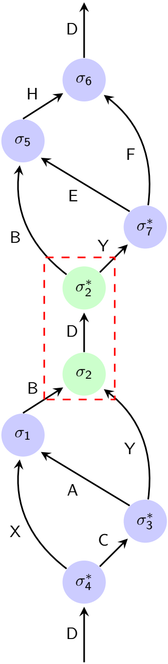


2-3 Move

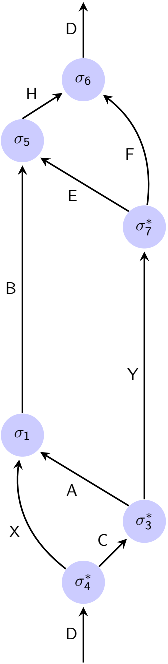



A
Y
B
E
C
D
X
F
H
A
Y
B
C
D
X
E
F
H
Y
D
B


σ3
Y
σ2
A
σ1
C
B
X
D
D
B
σ2
σ1
Y
A
X
C
σ3*
σ4*
D
σ4
σ7
F
σ6
E
σ5
Y
H
B
D
D
H
σ6
σ5
F
E
B
Y
σ7*
σ2*
D
σ2

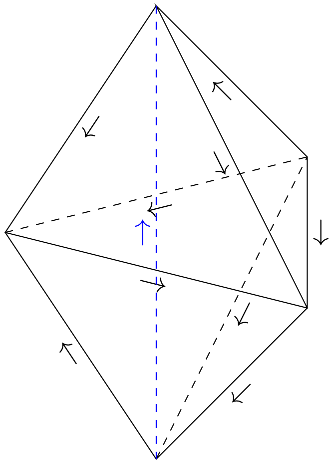
$\sum_{\sigma_2}$ 

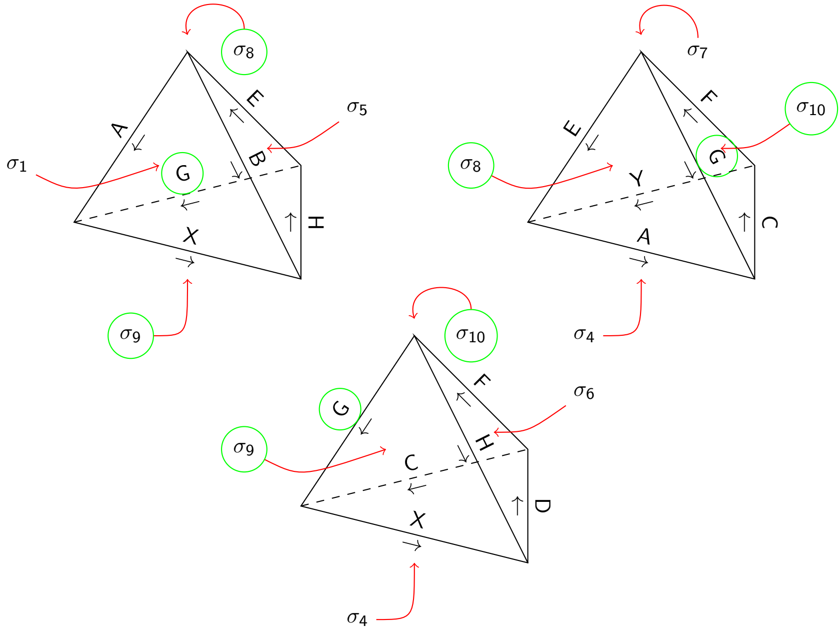
D
σ_6
H
F
σ_5
E
σ_7^*
B
Y
σ_2^*
D
σ_2
B
σ_1
Y
A
σ_3^*
X
C
σ_4^*
D

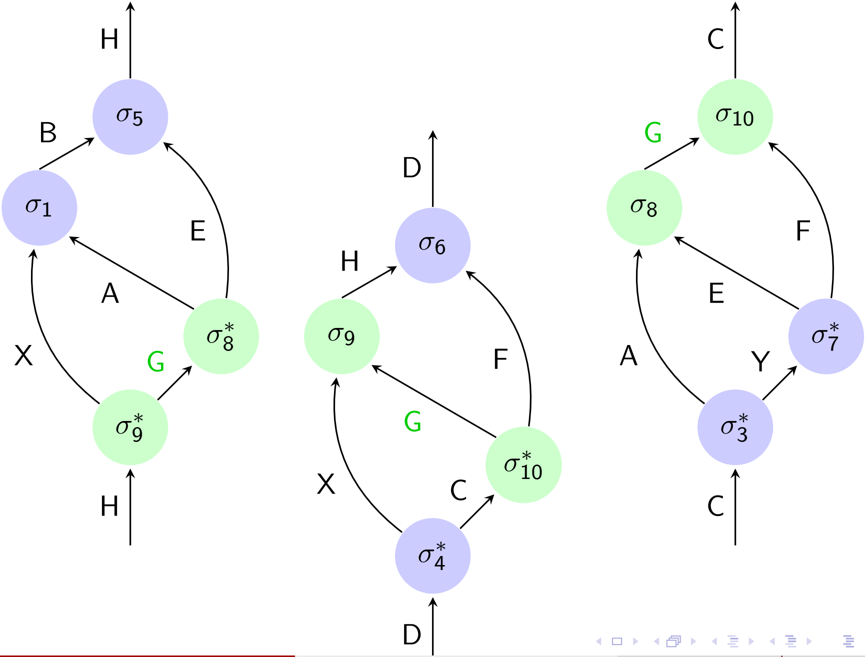


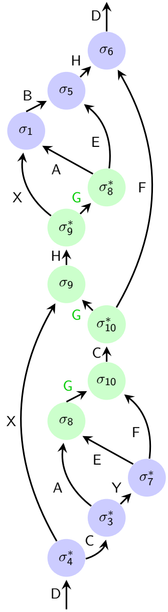
D
σ_6
H
F
σ_5
E
σ_7^*
B
Y
σ_1
A
σ_3^*
X
C
σ_4^*
D

Second Half of 2-3 Move




σ_8
A
E
σ_5
σ_1
G
B
X
H
σ_9
σ_7
F
σ_{10}
E
σ_8
G
Y
A
C
σ_4
σ_{10}
F
G
σ_6
σ_9
C
H
X
D
σ_4


H
σ_5
B
σ_1
E
A
σ_8^*
X
G
σ_9^*
H
D
σ_6
H
σ_9
F
G
σ_{10}^*
X
C
σ_4^*
D
C
σ_{10}
G
σ_8
F
E
σ_7^*
A
Y
σ_3^*
C


D
σ6
H
σ5
B
σ1
E
A
σ*8
G
F
X
σ*9
H
σ9
G
σ*10
C
σ10
G
X
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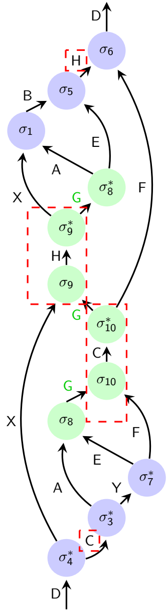
F
E
σ*7
A
Y
σ*3
C
σ*4
D


D
σ6
H
σ5
B
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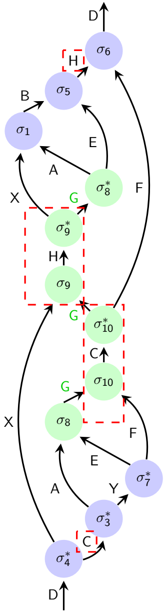
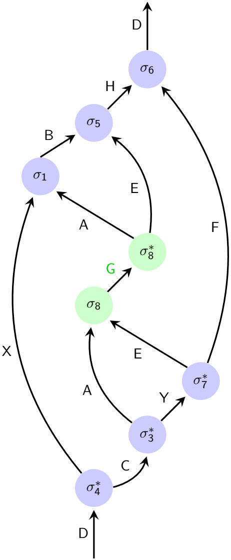
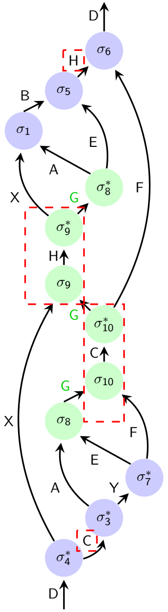
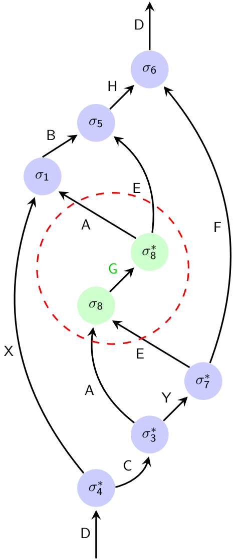
E
A
σ*8
F
X
G
σ*9
H
σ9
G
σ*10
C
σ10
G
σ8
X
F
E
σ*7
A
Y
σ*3
C
σ*4
D


D
σ6
H
σ5
B
σ1
E
A
σ8*
G
F
X
σ9*
H
σ9
G
σ10*
C
σ10
G
σ8
X
F
E
σ7*
A
Y
σ3*
C
σ4*
D

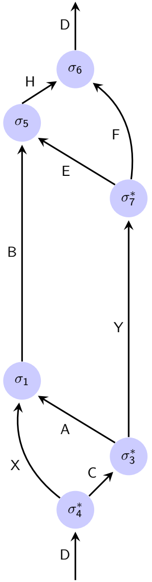

D
σ6
H
σ5
B
σ1
E
A
F
σ8*
G
σ8
X
E
σ7*
A
Y
σ3*
C
σ4*
D



D
σ_6
H
σ_5
B
σ_1
E
A
σ_8^*
F
X
G
σ_9^*
H
σ_9
G
σ_{10}^*
C
σ_{10}
G
σ_8
X
F
E
σ_7^*
A
Y
σ_3^*
C
σ_4^*
D



D
σ_6
H
σ_5
B
σ_1
E
A
F
σ_8^*
G
σ_8
X
E
σ_7^*
A
Y
σ_3^*
C
σ_4^*
D



D
σ_6
H
σ_5
F
E
σ_7^*
B
Y
σ_1
A
X
C
σ_3^*
σ_4^*
D

What about the correction factors?

- Correction factor of first half (2)

$$\sqrt{d(B)^{-1}d(C)^{-1}}\sqrt{d(H)^{-1}d(Y)^{-1}}$$

- Correction factor of second half (3)

$$\sqrt{d(B)^{-1}d(G)^{-1}}\sqrt{d(H)^{-1}d(C)^{-1}}\sqrt{d(G)^{-1}d(Y)^{-1}}$$

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$$\sqrt{d(B)^{-1}d(C)^{-1}}\sqrt{d(H)^{-1}d(Y)^{-1}}$$

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$$\sqrt{d(B)^{-1}d(G)^{-1}}\sqrt{d(H)^{-1}d(C)^{-1}}\sqrt{d(G)^{-1}d(Y)^{-1}}$$

They differ by $d(G)^{-1}$

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$$\sqrt{d(B)^{-1}d(G)^{-1}}\sqrt{d(H)^{-1}d(C)^{-1}}\sqrt{d(G)^{-1}d(Y)^{-1}}$$

They differ by $d(G)^{-1}$
Edge Correction Factor: $\prod_{\text{edges}} d(\text{edge label})$

$$Z(M, \underbrace{\mathcal{T}}_{\text{triangulation}}) = \sum_{\substack{\text{edge}\\\text{lables}}} \sum_{\substack{\text{face}\\\text{labels}}} \prod_{\substack{\text{tetrahedra}\\ T}} Z_{\text{cor}}(T) \prod_{\text{edges}} \left(\substack{\text{edge}\\\text{factor}}\right)$$

Invariance:

- Vertex order ✓
- 2-3 Pachner Move ✓
- 1-4 Pachner Move

What about the 1-4 Move?

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Exercise

Prove that the two sides of the 1-4 Move differ by a factor of

$$w = \sum_E d(E)^2$$

where E ranges over a complete set of simples.

The Turaev-Viro Invariant

$$Z(M, \underbrace{\mathcal{T}}_{\text{triangulation}}) = \frac{1}{w^{\#\text{vertices}}} \cdot \sum_{\substack{\text{edge}\\ \text{lables}}} \sum_{\substack{\text{face}\\ \text{labels}}} \prod_{\substack{\text{tetrahedra}\\ T}} Z_{\text{cor}}(T) \prod_{\text{edges}} \left(\begin{smallmatrix}\text{edge}\\ \text{factor}\end{smallmatrix}\right)$$

Invariance:

- Vertex order ✓
- 2-3 Pachner Move ✓
- 1-4 Pachner Move ✓

Exercise

For the A_2 planar algebra, for M^3 closed,

$$Z(M) = \#(H^1(M; \mathbb{Z}/2)).$$