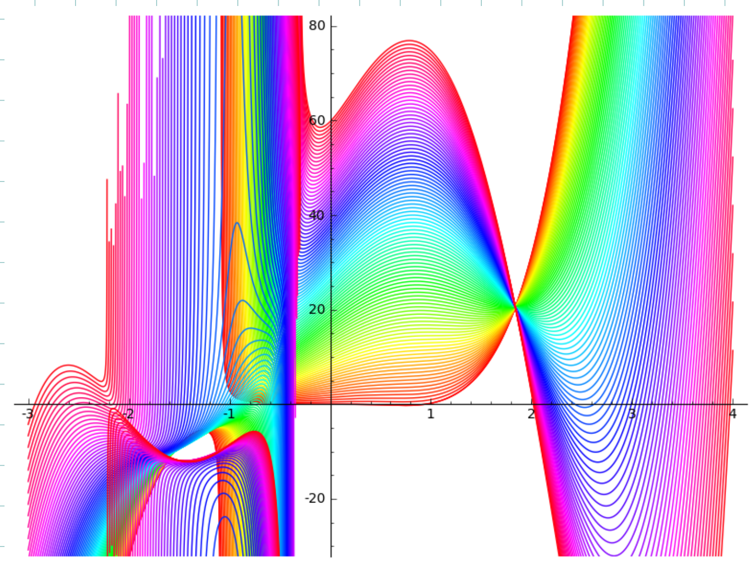
Algebraic Deformations of Rational Functions



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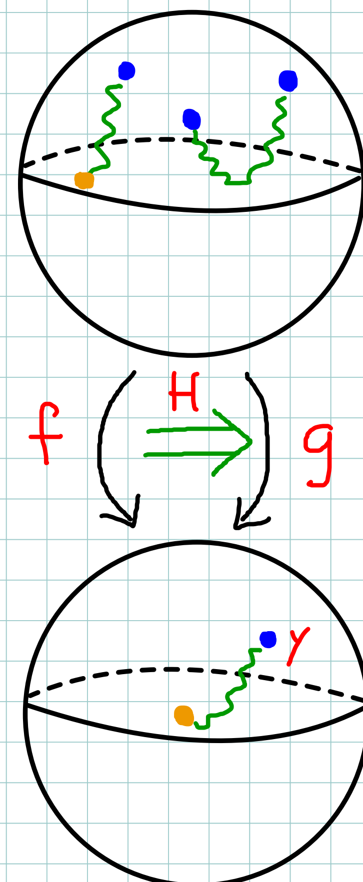

Kyle Ormsby
MIT
February 21, 2013

Outline:

$\boxed{\text{Algebraic puzzle}}$

Linking rational functions w/ polynomials

Outline:

$\boxed{\text{Algebraic puzzle}} \longrightarrow \boxed{\text{Topological avatar}}$

Linking rational functions w/ polynomials

Wrapping spheres around spheres

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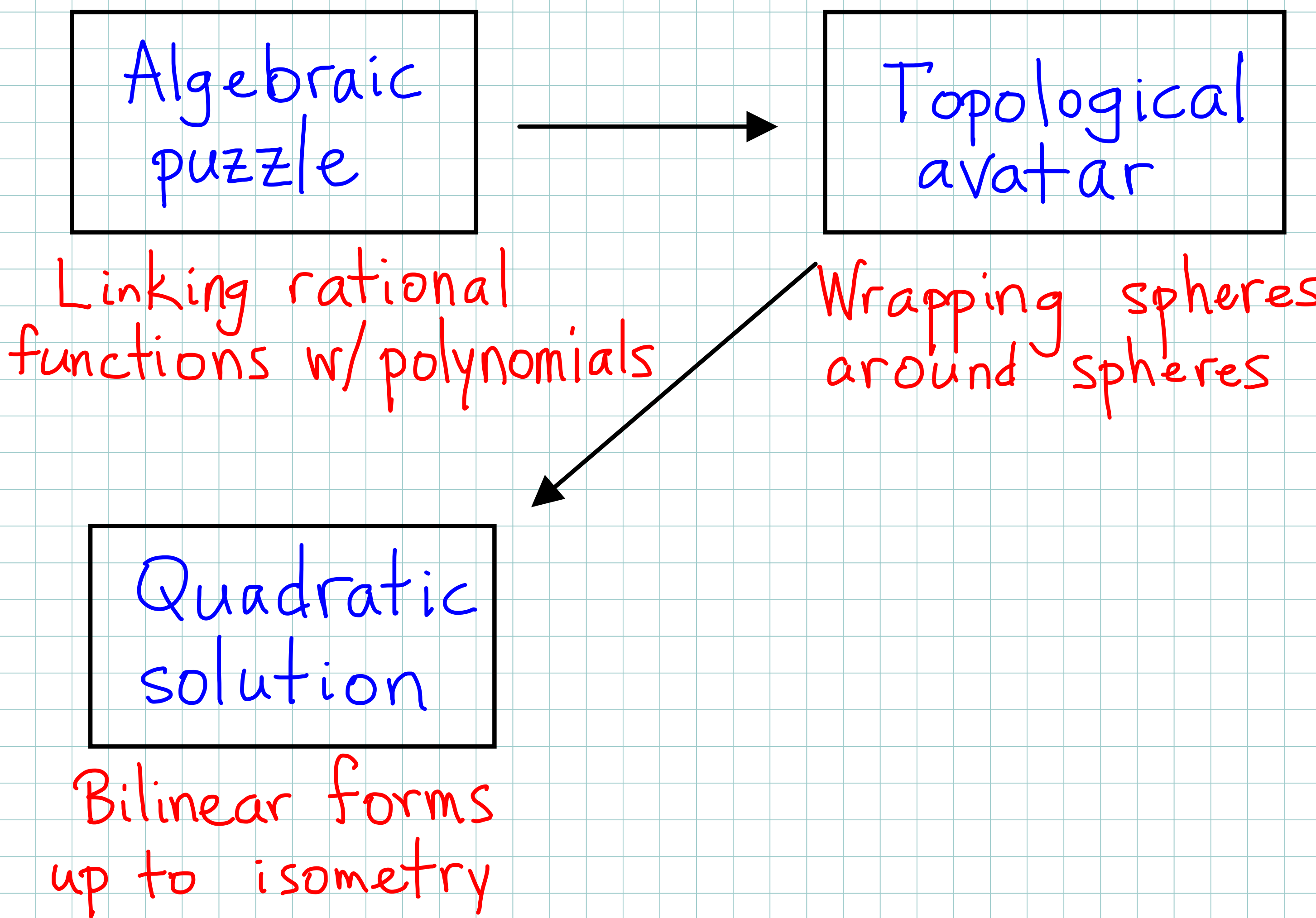
Algebraic puzzle
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Quadratic solution
Bilinear forms up to isometry

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Motivic homotopy

Where the wild things are

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A puzzle:

Rational functions

$$f = \frac{p}{q} = \frac{X^n + a_{n-1}X^{n-1} + \cdots + a_0}{b_{n-1}X^{n-1} + \cdots + b_0},$$

$$g = \frac{p'}{q'} = \frac{X^n + a'_{n-1}X^{n-1} + \cdots + a'_0}{b'_{n-1}X^{n-1} + \cdots + b'_0}.$$

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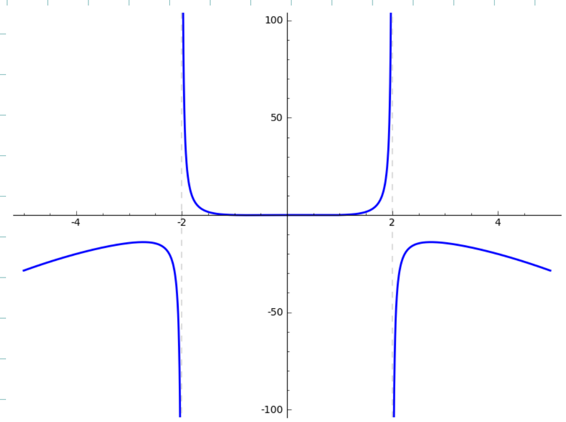
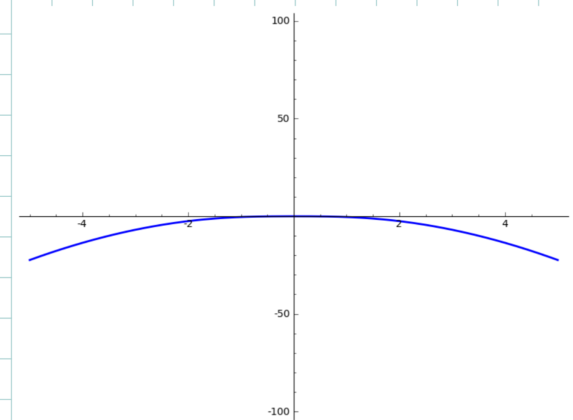
E.g. $f = \dfrac{X^4 - X^2}{-X^2+4}$ $\qquad g = \dfrac{X^4 + X^2}{-X^2 - 4}$





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When is there a new rational function

$$H(X,T) = \frac{P}{Q} = \frac{X^n + A_{n-1}(T)X^{n-1} + \cdots + A_0(T)}{B_{n-1}(T)X^{n-1} + \cdots + B_0(T)}$$

such that

$$H(X,0) = f, \quad H(X,1) = g \,?$$

[The $A_i(T)$ and $B_i(T)$ are polynomials.]

Refining the puzzle:

Assume all coefficients are real numbers.

For a rational function $H(X,T)$ and complex number t, let $H_t = H(X,t) = \frac{P_t}{Q_t}$.

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If $H_0 = f$ and $H_1 = g$, call H an

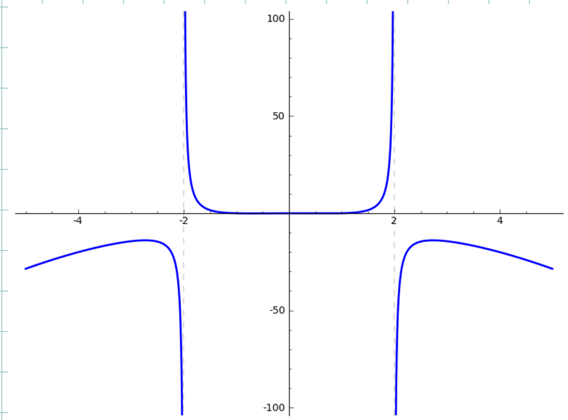
ALGEBRAIC DEFORMATION

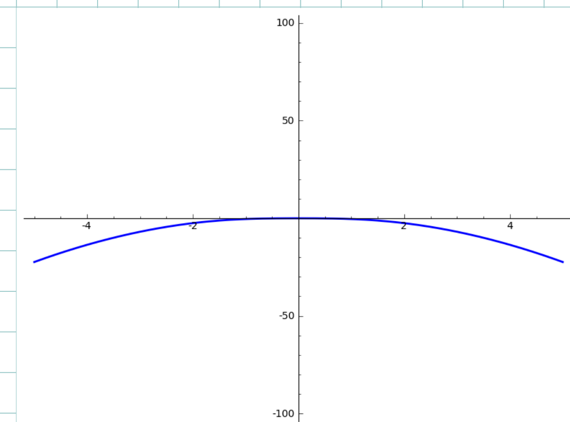
of f to g and write $H : f \simeq g$.

Example:

$$\frac{X^4 - X^2}{-X^2 + 4}$$

$$\frac{X^4 + X^2}{-X^2 - 4}$$





$$H(X,T) = \frac{X^4 + (2T-1)X^2 - 48T^2 + 48T}{-X^2 - 8T + 4}$$

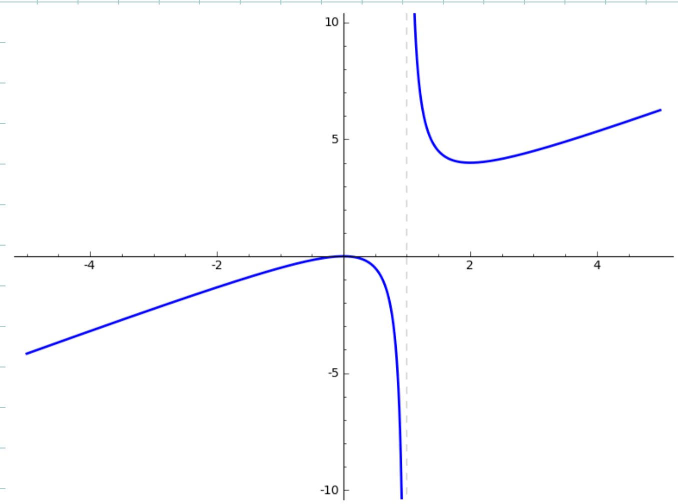
http://math.mit.edu/~ormsby/alg_def.gif

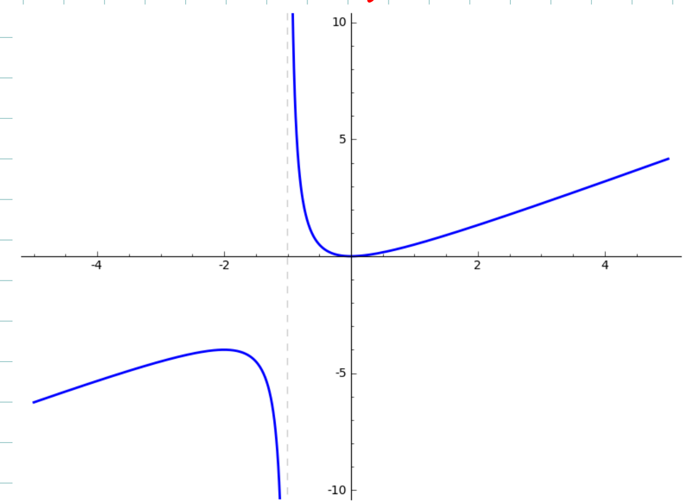
Non-example:

$$H(X,T) = \frac{X^2}{X+2T-1}$$

We have

$$H(X,0) = \frac{X^2}{X-1} \qquad H(X,1) = \frac{X^2}{X+1}$$





but...

http://math.mit.edu/~ormsby/not_def.gif

The final puzzle:

Classify rational functions

$$f = \frac{p}{q} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots + b_0}$$

up to ALGEBRAIC EQUIVALENCE.

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Two degree n rational functions f, g are algebraically equivalent if there is a chain of algbraic deformations

$$f = f_0 \underset{H_1}{\simeq} f_1 \underset{H_2}{\simeq} f_2 \simeq \cdots \underset{H_m}{\simeq} f_m = g$$

Maps of spheres:

Plugging real or complex values into $f = \frac{p}{q}$,

$$\left.\begin{array}{l} f_{\mathbb{R}} : \mathbb{R} \dashrightarrow \mathbb{R} \\ f_{\mathbb{C}} : \mathbb{C} \dashrightarrow \mathbb{C} \end{array}\right\} \text{NOT defined when } q(x) = 0.$$

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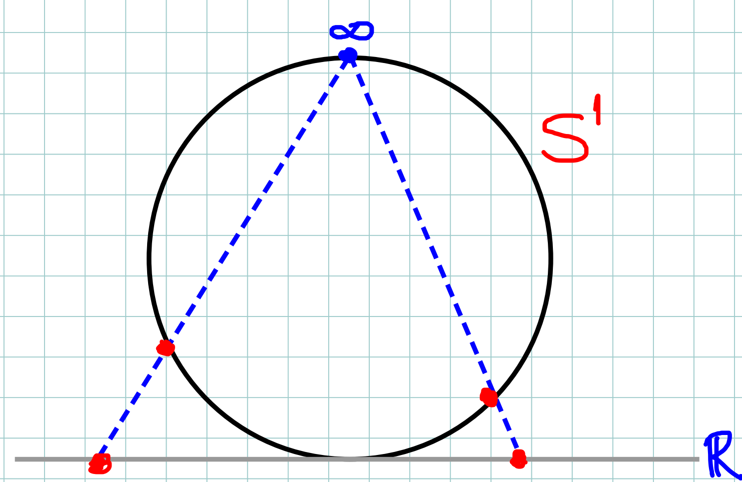

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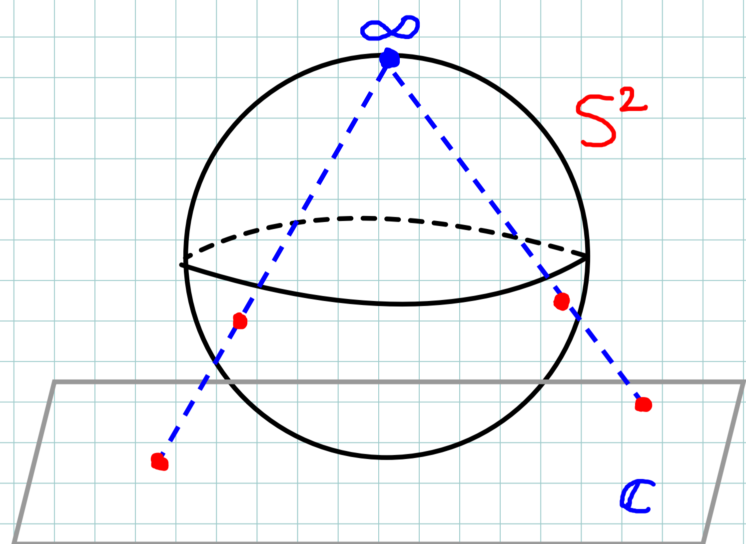

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Maps of spheres (ct'd):

So real rational functions produce continuous maps

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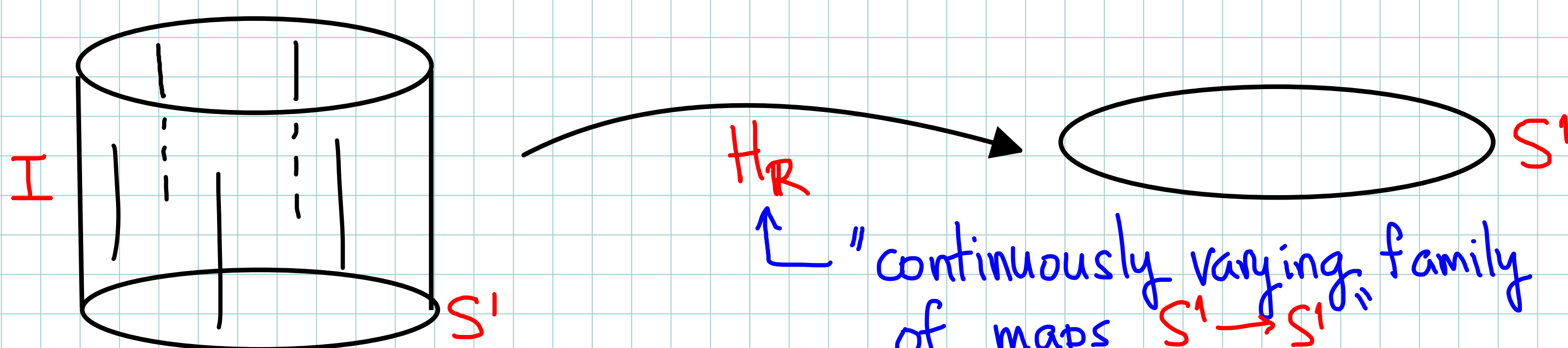
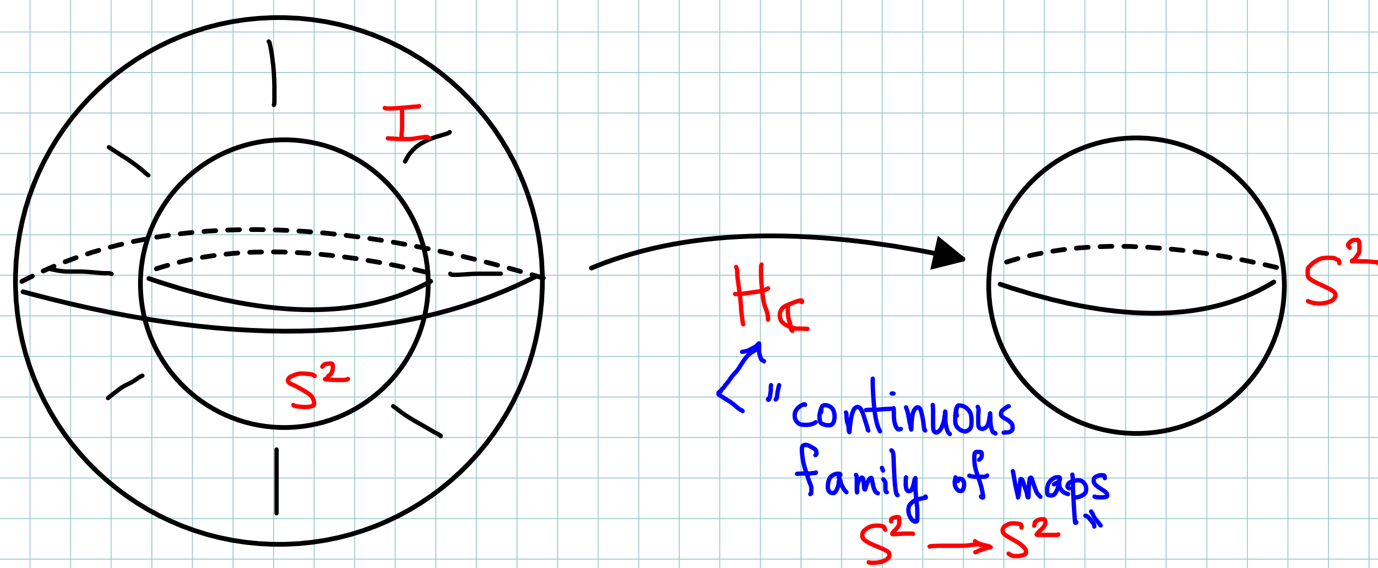
On "real points" we can view $H_{\mathbb{R}}$ as



"continuously varying family of maps $S^1 \to S^1$"

A topological variation:

On "complex points" we get



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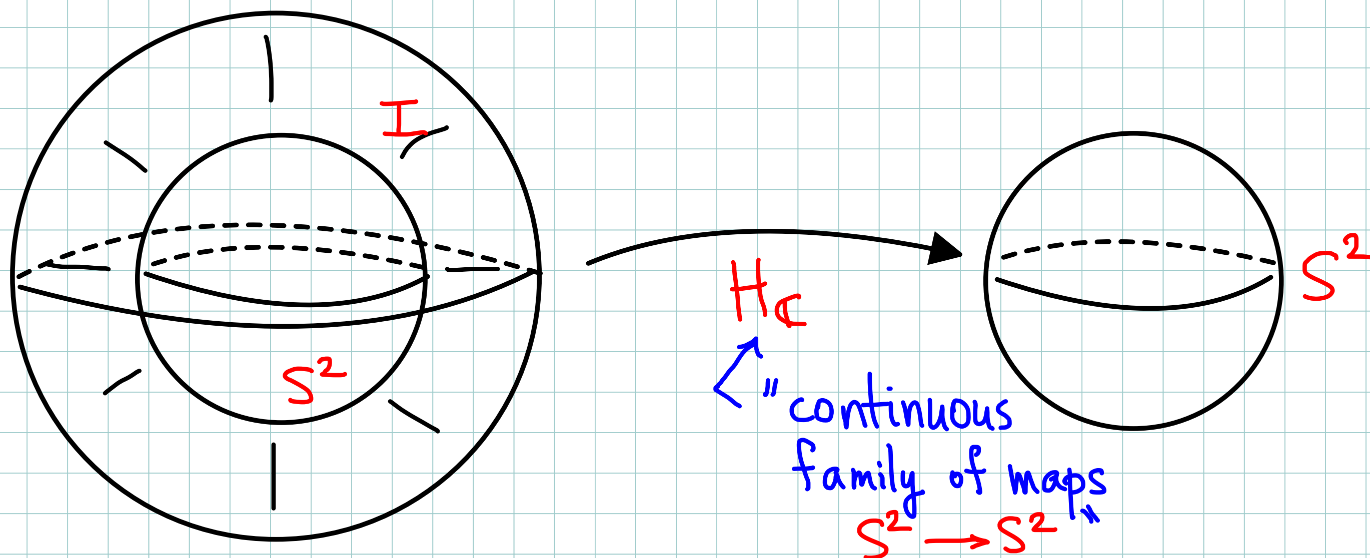

$\alpha, \beta : X \to Y$ are HOMOTOPIC if there is a HOMOTOPY $H : X \times I \to Y$ such that $H_0 = \alpha$, $H_1 = \beta$.

An algebraic deformation induces homotopies

$$H_{\mathbb{R}} : f_{\mathbb{R}} \simeq g_{\mathbb{R}}, \quad H_{\mathbb{C}} : f_{\mathbb{C}} \simeq g_{\mathbb{C}}$$

Wrapping spheres around spheres:

Let's address the simpler problem of classifying continuous maps $S^1 \to S^1$ and $S^2 \to S^2$ (that send ∞ to ∞) up to homotopy.

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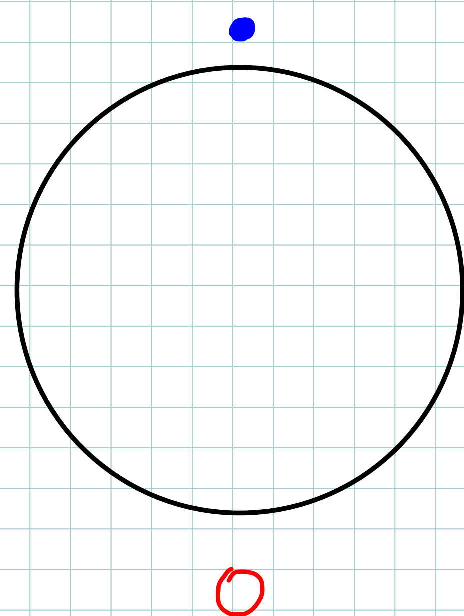

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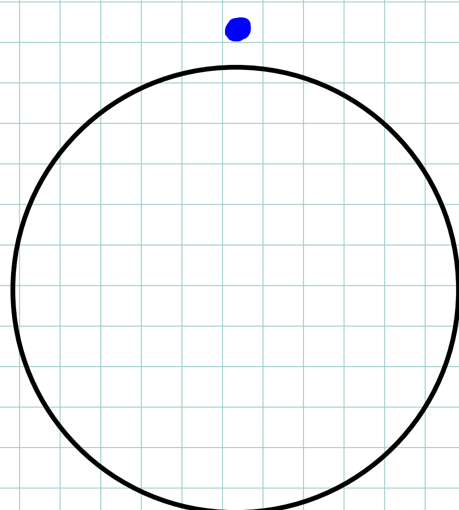

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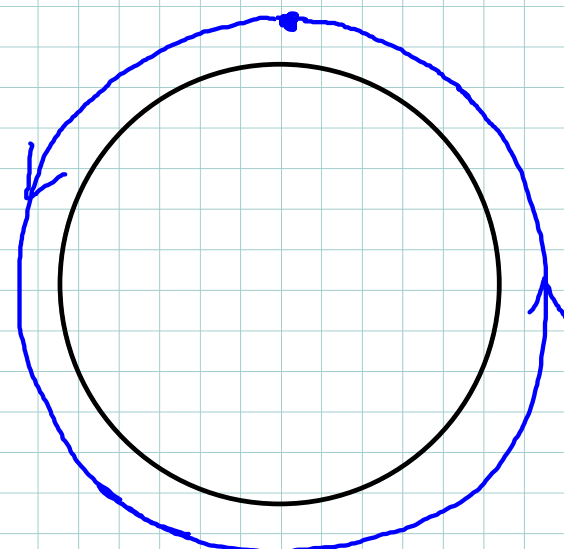

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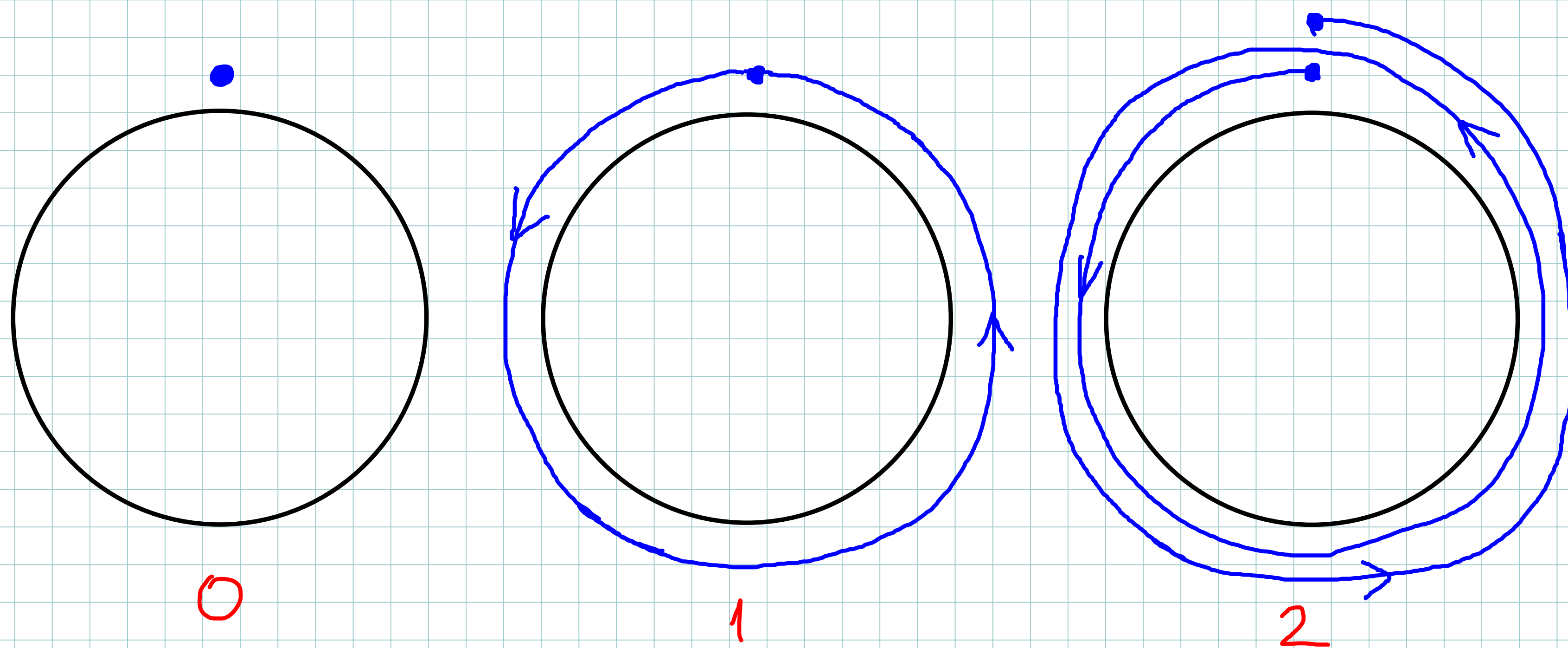

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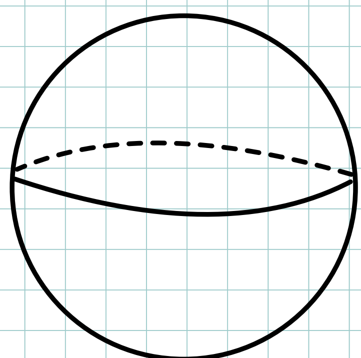

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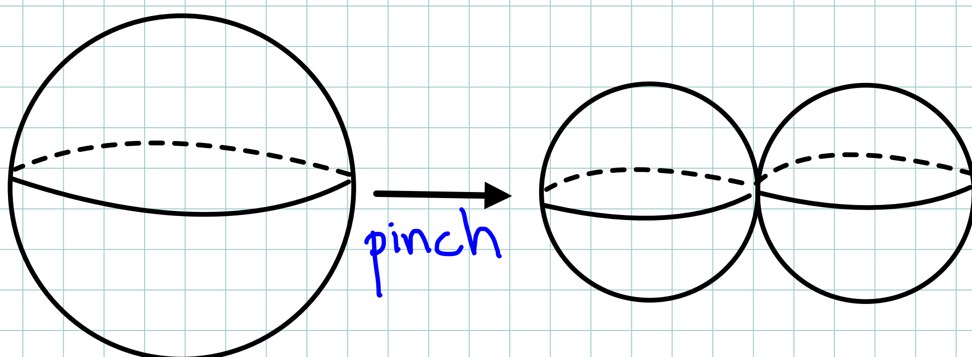

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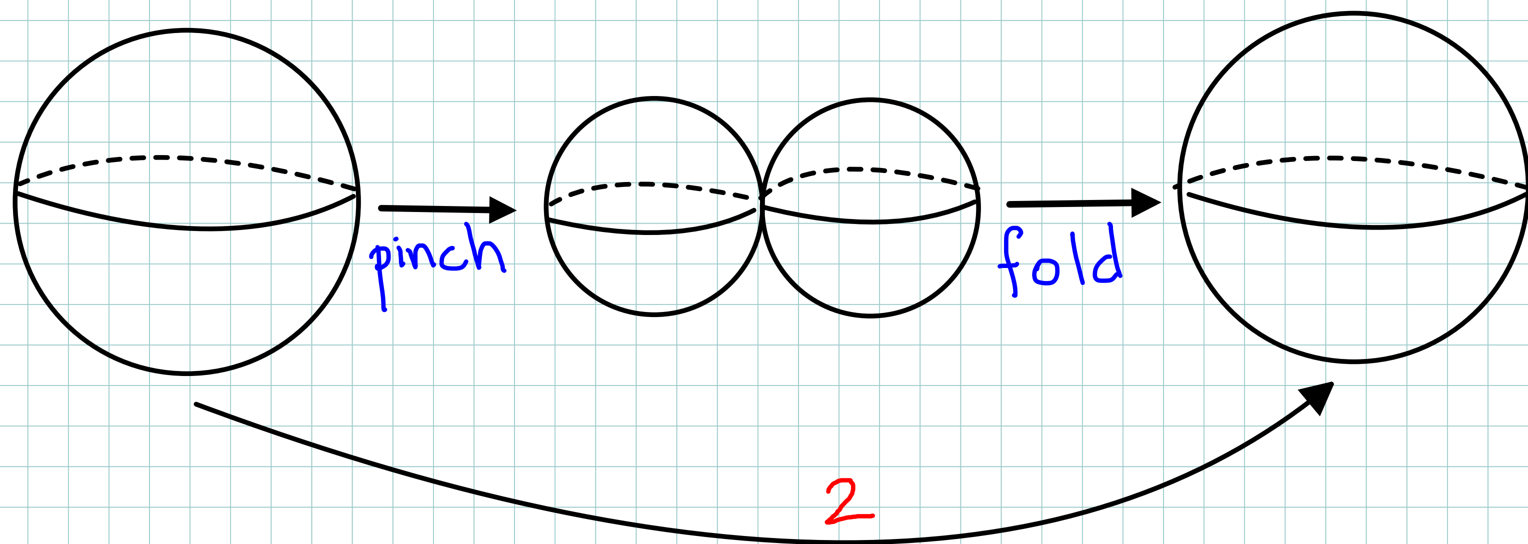
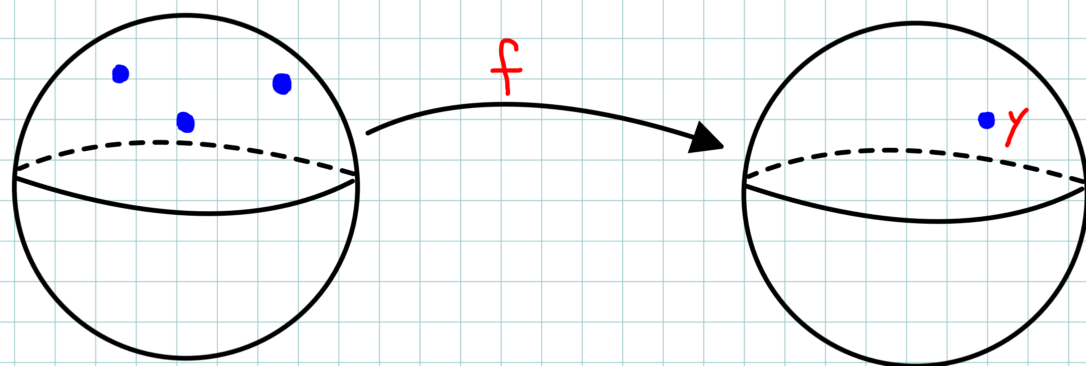


The degree of a map:

We may assume $f : S^n \to S^n$ is smooth and choose a regular value y in its target.



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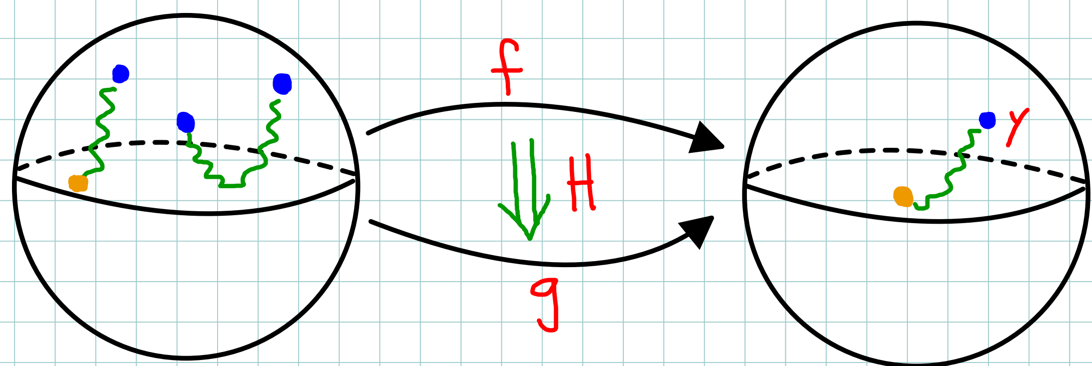

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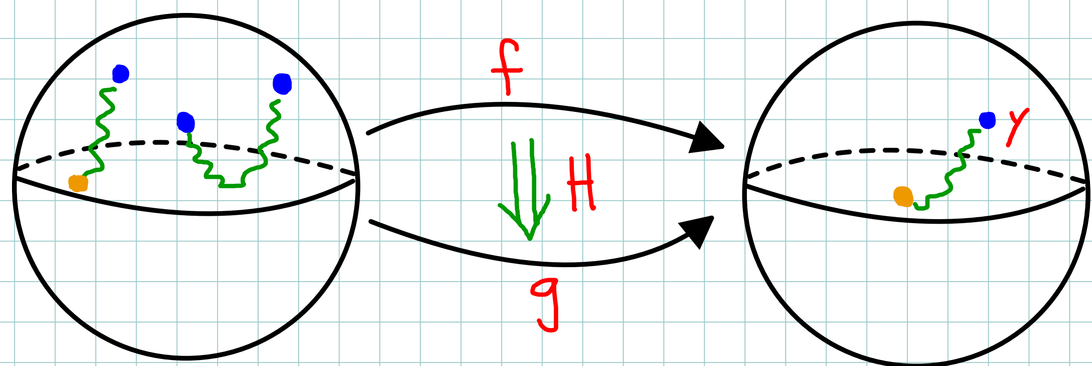

If $f^{-1}(y) = \{x \in S^n \mid f(x) = y\}$, then $|f^{-1}(y)|$ changes by an <u>even</u> number under homotopy.

The value $|f^{-1}(y)| \bmod 2$ is called the MOD 2 DEGREE of f.

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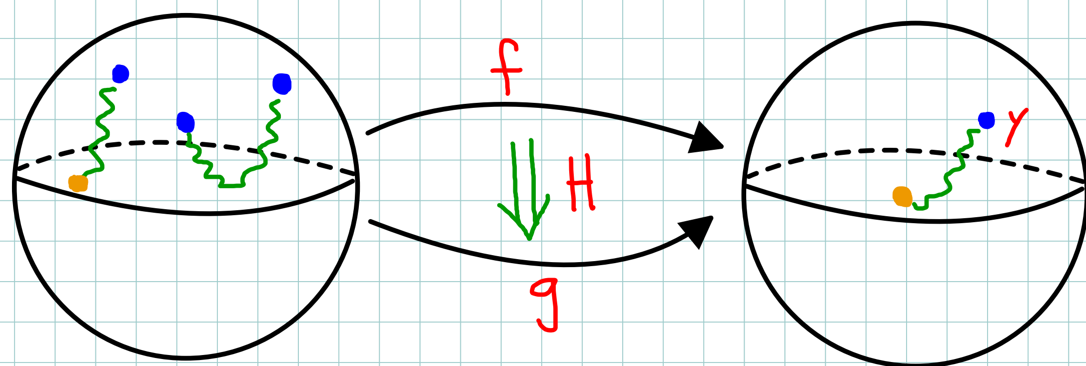

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Can we do better?

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Label each point in $f^{-1}(y)$ with a sign:

$+1$ if f locally PRESERVES ORIENTATION

-1 if f locally REVERSES ORIENTATION

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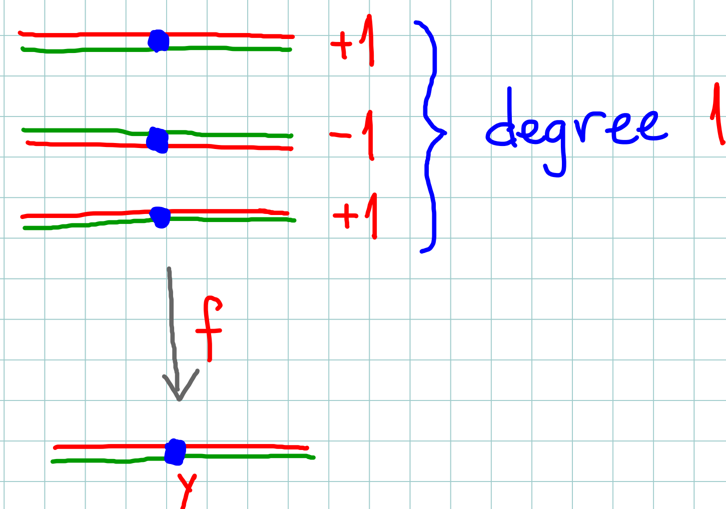

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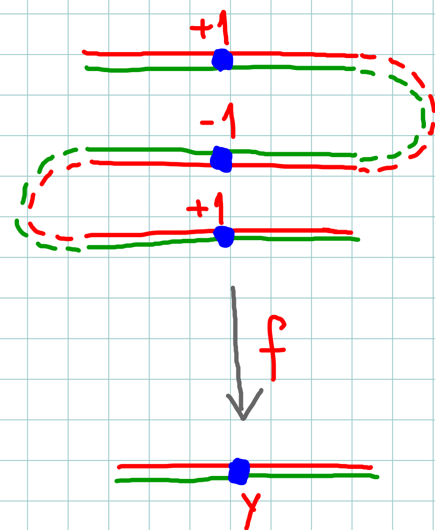

Homotopies only cancel points with opposite sign, so the sum of the signs in $f^{-1}(y)$ is invariant under homotopy: the DEGREE of f, $\deg(f)$.

Brouwer's Theorem:

The degree map is a bijective correspodence between homotopy classes of maps $S^n \to S^n$ and the integers, $\mathbb{Z}$, for $n \geq 1$.

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<u>Consequence</u>: If f and g are algebraically equivalent rational functions, then

$$\deg(f_{\mathbb{R}}) = \deg(g_{\mathbb{R}}) \text{ and } \deg(f_{\mathbb{C}}) = \deg(g_{\mathbb{C}})$$

<u>Question</u>: What algebraic data determines $\deg(f_{\mathbb{R}})$ and $\deg(f_{\mathbb{C}})$?

Degrees of rational functions:

Proposition: If $f = \frac{P}{Q} = \frac{X^n + a_{n-1}X^{n-1} + \cdots + a_0}{b_{n-1}X^{n-1} + \cdots + b_0}$ is a degree n rational function, then $\deg(f_c) = n$.

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<u>Proof sketch</u>: Consider $f_{\mathbb{C}}^{-1}(0)$. We have

$$|f_{\mathbb{C}}^{-1}(0)| = |P^{-1}(0)| = |\{\text{roots of } P\}|.$$

If 0 is a regular value of $f_{\mathbb{C}}$ then this number is n.

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EXERCISE: $f_{\mathbb{C}}$ preserves orientation everywhere.

$\square$

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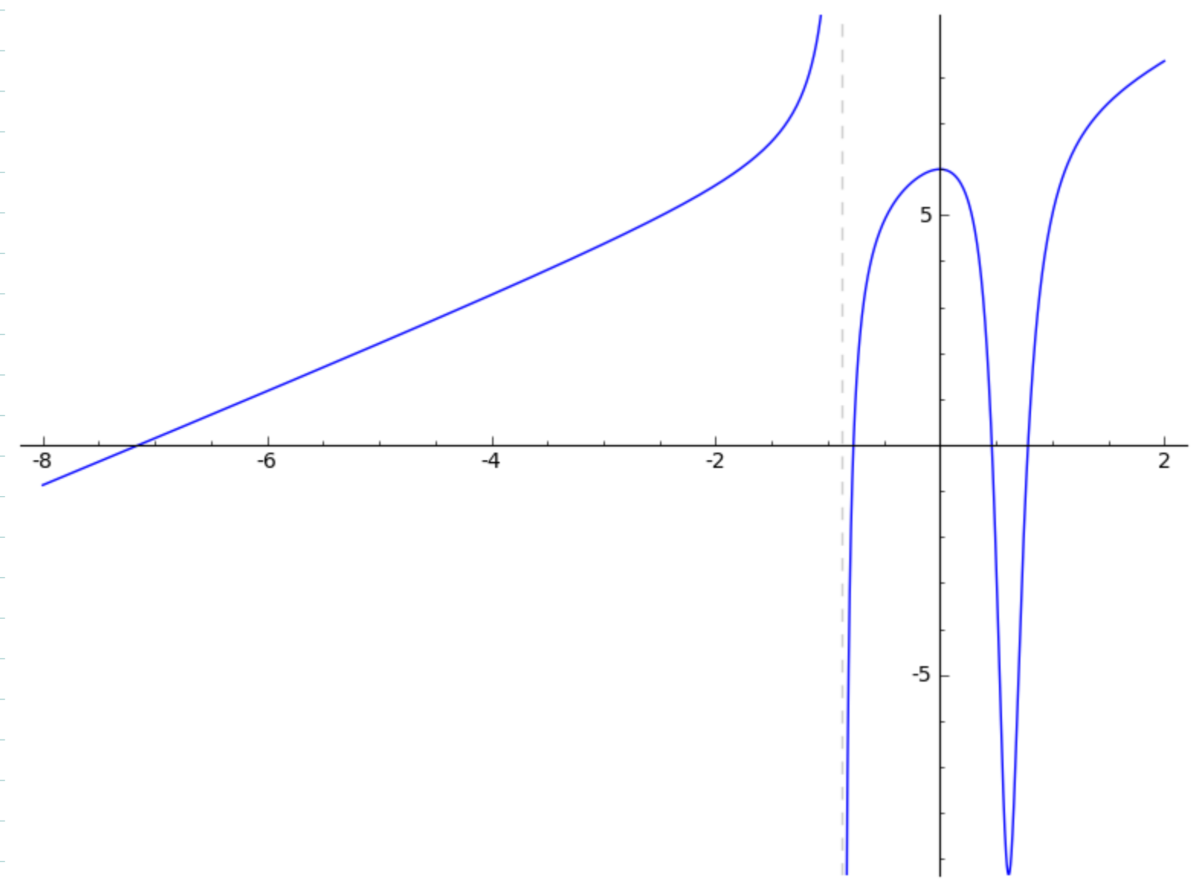

Proposition: If $f'(x) \neq 0$, $\deg(f_{\mathbb{R}}) = \sum_{f(x)=0} \operatorname{sign}(f'(x))$. $\square$

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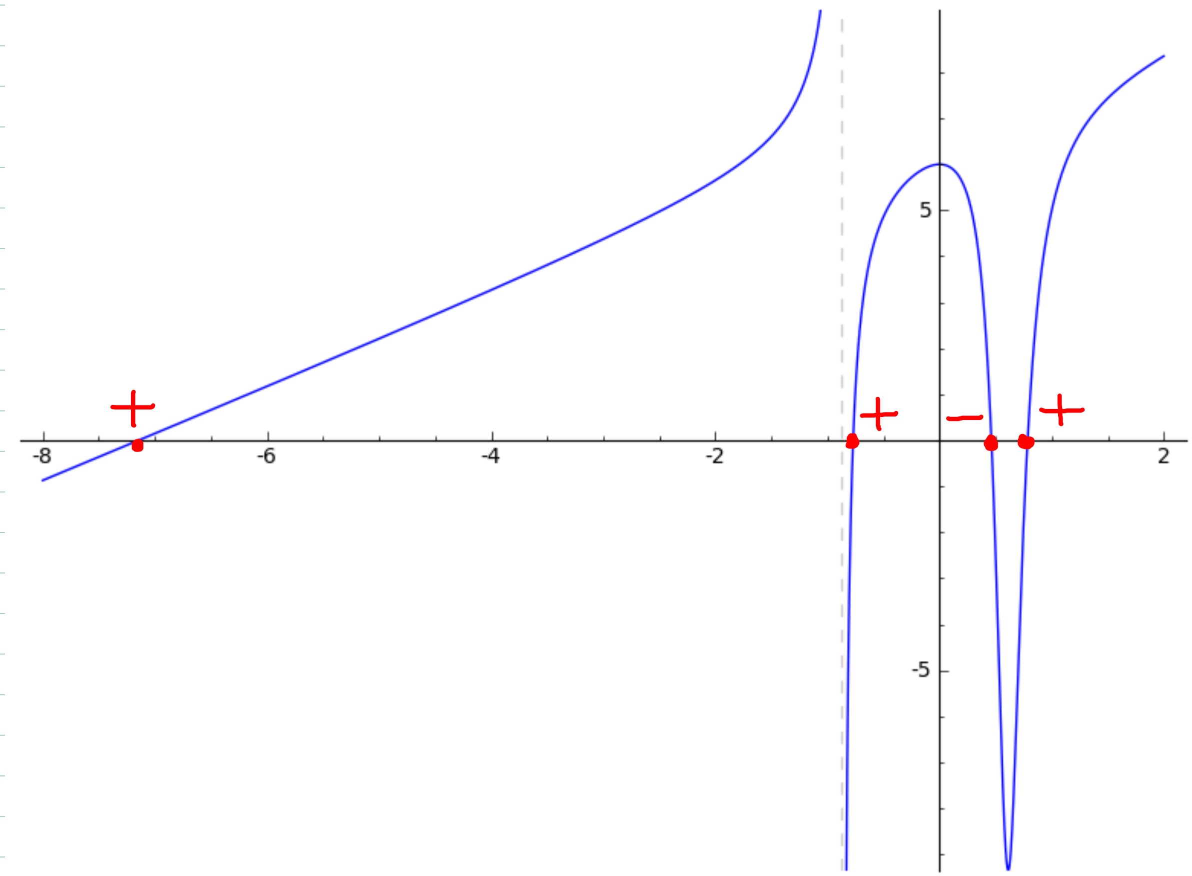

$$\deg(f_{\mathbb{R}}) = 1+1-1+1 = 2$$

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A relationship between degrees:

If f is a degree n rational function, we know that $\deg(f_{\mathbb{C}}) = n$.

How does $\deg(f_{\mathbb{R}})$ compare to $\deg(f_{\mathbb{C}})$?

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Observe:
- f has at most n real roots
- non-real roots come in conjugate pairs
- $\deg(f_{\mathbb{R}}) = n - (\#\text{ non-real roots}) - 2(\#\text{ roots } x \text{ w/ } f'(x) < 0)$

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- $\deg(f_{\mathbb{R}}) = n - (\#\text{ non-real roots}) - 2(\#\text{ roots } x \text{ w/ } f'(x)<0)$

Proposition: $-n \leq \deg(f_{\mathbb{R}}) \leq n$ and $\deg(f_{\mathbb{R}}) \equiv n \ (2)$.

Summary:

If f and g are algebraically equivalent, we know that $\deg(f_{\mathbb{R}}) = \deg(g_{\mathbb{R}})$ and

$$\deg(f_{\mathbb{C}}) = \deg(g_{\mathbb{C}}).$$

We also know $n = \deg(f_{\mathbb{C}})$ is a non-negative integer and $m = \deg(f_{\mathbb{R}})$ satisfies $-n \leq m \leq n$, $m \equiv n \ (2)$.

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We also know $n = \deg(f_{\mathbb{C}})$ is a non-negative integer and $m = \deg(f_{\mathbb{R}})$ satisfies $-n \leq m \leq n$, $m \equiv n \ (2)$.

Q1 Are all such pairs (m, n) realized as $(\deg(f_{\mathbb{R}}), \deg(f_{\mathbb{C}}))$ for some f?

Q2 Does $(\deg(f_{\mathbb{R}}), \deg(f_{\mathbb{C}})) = (\deg(g_{\mathbb{R}}), \deg(g_{\mathbb{C}}))$ imply f is algebraically equivalent to g?

The resultant:

Polynomials $p = X^n + a_{n-1} X^{n-1} + \cdots + a_0$,

$$q = b_{n-1} X^{n-1} + \cdots + b_0$$

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E.g. If $p = X^3 + 5X^2 + 7X - 3$ and

$q = 2X^2 - X + 11$, then

$$\mathrm{res}(p,q) = \det \left(\begin{array}{cc|ccc} 1 & & 2 & & \\ 5 & 1 & -1 & 2 & \\ 7 & 5 & 11 & -1 & 2 \\ -3 & 7 & & 11 & -1 \\ & -3 & & & 11 \end{array} \right)$$

Sylvester matrix

The resultant:

If $H = \frac{P}{Q} : f \simeq g$ then we can form $\mathrm{res}(P,Q)$ as a polynomial in T.

In order for H to be an algebraic deformation, $\mathrm{res}(P,Q)$ must be constant and nonzero.

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So if $f = \frac{p}{q}$, $g = \frac{p'}{q'}$, then

$$\mathrm{res}(p, q) = \mathrm{res}(P, Q) = \mathrm{res}(p', q').$$

Let $\mathrm{res}(f) = \mathrm{res}(p, q)$.

The solution:

<u>Notation</u> $\mathcal{F}$ = pointed rational functions

$\pi_0\mathcal{F}$ = pt'd rat'l f'ns up to algebraic equivalence

$$\boxed{\begin{aligned} \pi_0\mathcal{F} &\xrightarrow{\cong} \left\{ (n,m,\lambda) \;\middle|\; \begin{array}{l} n\in\mathbb{N},\ m\in\mathbb{Z},\ \lambda\in\mathbb{R} \\ -n\le m\le n,\ n\equiv m\ (2), \\ (-1)^{(n^2-m)/2}\lambda > 0 \end{array} \right\} \\ [f] &\longmapsto \left(\deg(f_{\mathbb{C}}), \deg(f_{\mathbb{R}}), \operatorname{res}(f)\right) \end{aligned}}$$

A generalization:

What if the coefficients of our functions were in

- $\mathbb{C}$
- $\mathbb{Q}$
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Q Could we still classify rational functions up to algebraic equivalence?

A generalization:

What if the coefficients of our functions were in

- $\mathbb{C}$ — just use $\deg(f_{\mathbb{C}})$ and $\operatorname{res}(f)$
- $\mathbb{Q}$ — current methods give partial results
- $\mathbb{Z}/p\mathbb{Z}$ — subtle
- some other field — a uniform answer?

instead of $\mathbb{R}$?

$\underline{\underline{Q}}$ Could we still classify rational functions up to algebraic equivalence?

The Bezout form:

For polynomials $p(X)$, $q(X)$, we have

$$X - Y \mid p(X)q(Y) - p(Y)q(X).$$

Hence

$$\delta_{p,q}(X,Y) = \frac{p(X)q(Y) - p(Y)q(X)}{X - Y}$$

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The BEZOUT FORM of $f=\frac{p}{q}$ is the bilinear form

$$\mathrm{Bez}(f)(\underline{x},\underline{y}) = \sum_{1\le k,\ell\le n} c_{k,\ell}\, x_k y_\ell .$$

with matrix $(c_{k,\ell})$.

Bezout form as derivative:

The Bezout form of f is closely related to f'.

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- If we're working over $\mathbb{R}$, this quantity is closely related to $\deg(f_{\mathbb{R}})$.
- We should think of $\mathrm{Bez}(f) = (c_{k,\ell})$ as an algebraic replacement for differential data.

A solution over any field:

Two bilinear forms B, B' are ISOMETRIC if there is an invertible matrix A such that $A^T B A = B'$.

Let $Bil(k)$ = isometry classes of bilinear forms over the field k.

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Let $\mathrm{Bil}(k)$ = isometry classes of bilinear forms over the field k.

<u>Theorem</u> (Cazanave, Morel, Barge, Lannes)
Algebraic equivalence classes of rat'l fns are completely determined by Bez and res.
There is precisely one for each isometry class B and scalar $\lambda \in k^\times$ such that $(-1)^{n(n-1)/2} \lambda \det(B) \in (k^\times)^2$.

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  classified by dimension and signature

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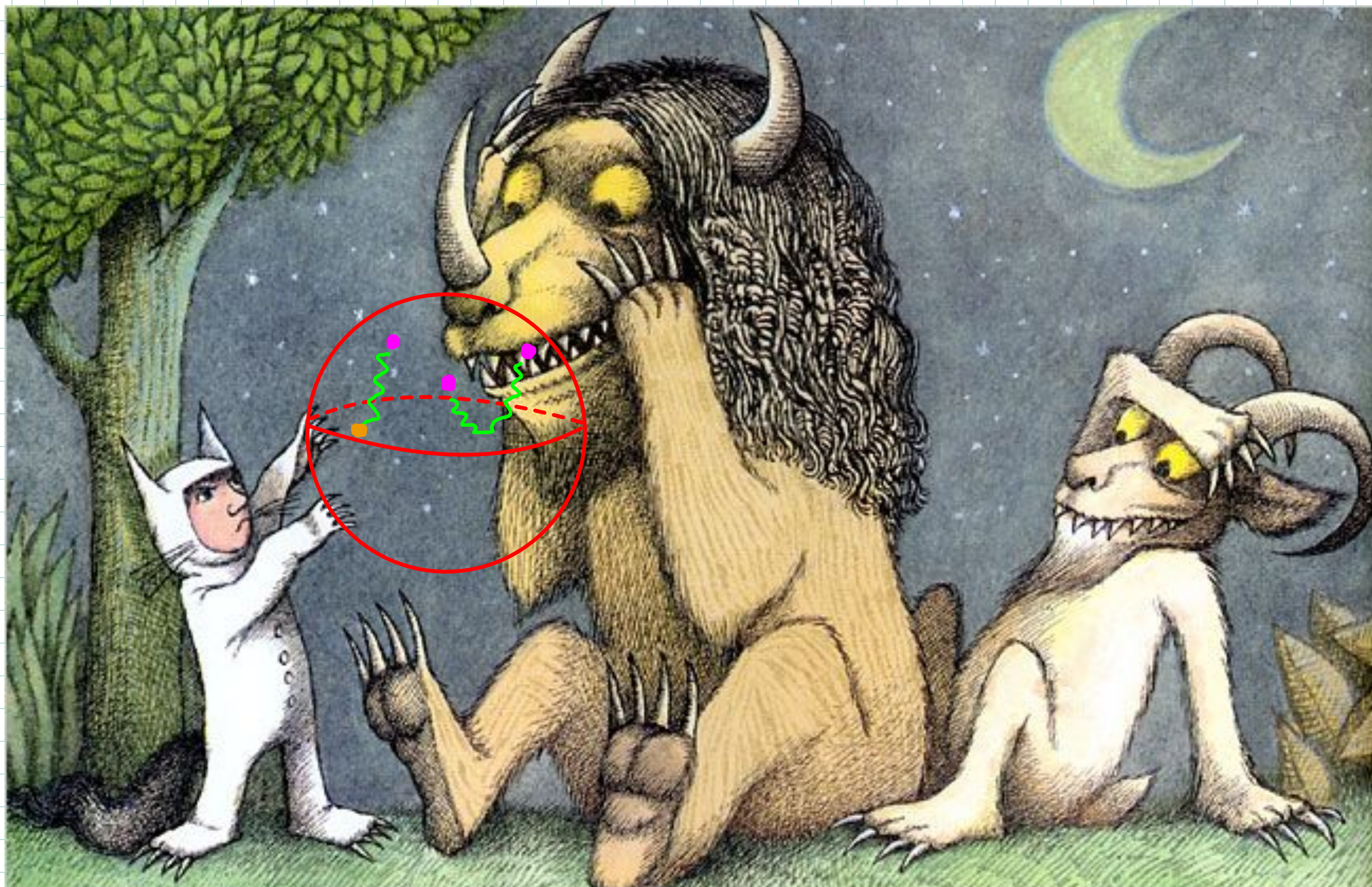
- $\mathrm{Bil}(\mathbb{Z}/p\mathbb{Z})$ [or $\mathrm{Bil}(\mathbb{F}_q)$]
  only two isometry classes in each dimension
- Bil(general field) — long and interesting history

Motivic Homotopy:



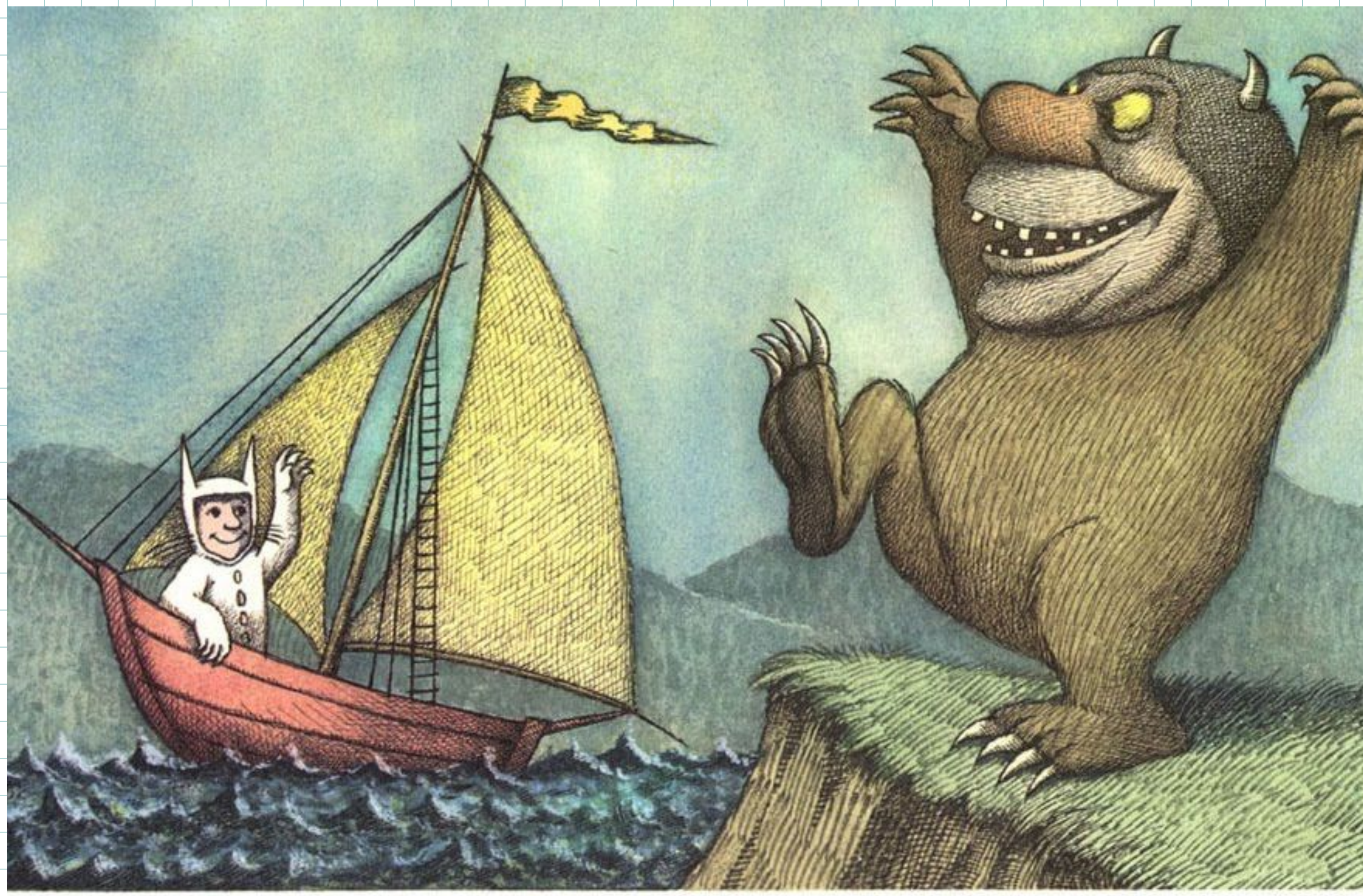
Motivic Homotopy:

- Rational functions are really $\mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$.

- Alg. def's are really $\mathbb{P}^1_k \times \mathbb{A}^1_k \longrightarrow \mathbb{P}^1_k$
  [naïve $\mathbb{A}^1$-homotopies].

- With enough ALGEBRAIC TOPOLOGY and ALGEBRAIC GEOMETRY applied, we have a brand new tool that answers decades-old questions about bilinear forms.

Thank you!



Slides and animations available at
http://math.mit.edu/~ormsby/