MATH 412: TOPICS IN ALGEBRA
FINAL HOMEWORK

Instructions: Select and solve three of the following homework problems written by your peers and related to their final projects. (You may not select your own problem.) Turn in your solutions at the start of the final exam: 18 December, 9A.M.–12P.M. in Eliot 207.

Problem 1 (Evan). Do one of the following two problems:

(a) Show inductively (without making use of the fact that Pfister forms are group forms) that if $x \in \mathsf{k}^\times$ is a sum of 2^n squares, then

$$2^n \langle 1 \rangle \cong 2^n \langle x \rangle$$

and, from this, deduce that $2^n \langle 1 \rangle$ is a group form.

(b) Let $\mathsf{k} = \mathbb{R}(\!(x)\!)$. Verify that $|\mathsf{k}^\times / \mathsf{k}^\boxtimes| = 4$ and that k is Pythagorean in order to conclude that $\mathrm{W}(\mathsf{k}) = \mathbb{Z}[G]$ where G is the group with elements $\{1, x\}$.

Problem 2 (Yunjia). Do one (or a few if you feel like it) of the following six problems on Clifford algebras. The number of stars indicates difficulty.

(a) A nice warm-up ($\star$):

As shown in section 0, there is a hidden subalgebra $\mathbb{C}$ inside $C(\mathbb{R}^2, |v|^2)$. But actually, $\mathbb{C}$ itself is a Clifford algebra. Find the quadratic form that generates the "honest" complex number. What is the quadratic space V in this case? The exterior algebra $\bigwedge(V)$ over an n-dimensional vector space V is an anticommutative algebra such that for any two basis vectors $\mathbf{e}_i$, $\mathbf{e}_j$, we have

$$0 = \{\mathbf{e}_i, \mathbf{e}_j\} = \mathbf{e}_i \wedge \mathbf{e}_j + \mathbf{e}_j \wedge \mathbf{e}_i,$$

in which $\wedge$ denotes the bilinear multiplication over the algebra. Given an n-dimensional vector space $V = \mathsf{k}^n$, argue that its exterior algebra $\bigwedge(V)$ is a Clifford algebra by finding its corresponding quadratic form.

(b) Does it work in 3-dimensional space? ($\star$):

Find $C(\mathbb{R}^3, |\mathbf{v}|^2)$ with the standard Euclidean metric $|\cdot|^2 = \langle 1, 1, 1 \rangle$. What is its even subalgebra $C^+(\mathbb{R}^3, |\mathbf{v}|^2)$?

(c) From quadratic space to symmetric bilinear space ($\star\star$):

Given a quadratic space (V, q), we know that by polarizing q, we obtain a (bijective) symmetric bilinear space (V, B). Therefore, it is quite natural to realize that there is an equivalent definition of the Clifford algebra over a symmetric bilinear space (V, B). How would you modify the definition of a Clifford algebra to make it compatible over (V, B)? [*Hint:* Some hinted structure from eq. (12) of my note may be helpful.]

(d) A quick corollary in class ($\star\star$):

Show the following proposition.

Proposition 1. *For a quadratic space V with two equivalent quadratic forms $q \cong q'$, their Clifford algebra $C(V, q) \cong C(V, q')$.*

[*Hint:* One may find the universal property of Clifford algebras very handy, and also to find an isomorphism is to find an inverse for a known homomorphism.]

(e) A combinatorial game ($\star$):
Show the following proposition.

Proposition 2. $\dim C(V, q) \leq 2^{\dim V}$.

[*Hint:* You probably want to learn something about the basis of Clifford algebras (Theorem 2.4 of my note), and recall that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.]

(f) A weird involution ($\star$):
Show the following proposition.

Proposition 3. *The canonical involution is an automorphism (an isomorphic endomorphism) over $C(V, q)$.*

[*Hint:* Proposition 1 may be helpful.]

Problem 3 (Henry). Suppose $\mathcal{K} = \mathbb{Q}(i)$ where $i := \sqrt{-1}$. Then consider the ideal $\mathfrak{a}$ in $\mathcal{O}_{\mathcal{K}}$ (see definition 2.5) given by

$$\mathfrak{a} := [i, 2 + i].$$

Find the quadratic form that *belongs* to $\mathfrak{a}$ (see definition 2.16 and example 2.3).

Problem 4 (Francis). Do one of the following two problems:

(a) Prove that for all field k, $\mathrm{rad}(\mathrm{W}(\mathsf{k})) = \mathrm{nil}(\mathrm{W}(\mathsf{k}))$.
(b) For k a formally real field, show that the Jacobson radical of $\mathrm{W}(\mathsf{k})$ is a prime ideal if and only if k has a unique orderings. (*Hint*: $\mathrm{rad}(\mathrm{W}(\mathsf{k})) = \bigcap \mathrm{MaxSpec}(\mathrm{W}(\mathsf{k}))$.)

Problem 5 (Luke). (a) Let $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, where the λ_i's are generators of k_nF. Show that if $\alpha_n(\lambda) = 0 \in I^n/I^{n+1}$, then $\lambda = 0 \in k_nF$. (Hint: Use Theorem 3.12.)
(b) For a given integer n, suppose that every element of k_nF can be expresed as a sum of three generators. Show that α_n is an isomorphism.

Problem 6 (Usman). Let $f = \frac{X^2 - \frac{3}{5}}{5X}$ and $k = \mathbb{R}$.

(a) Find the decomposition of f into monic polynomials. Without computation determine the symmetric bilinear form that f is mapped to under the Bézout map.
(b) Compute the Bézout form for f by hand.

Problem 7 (Riley). Let q be a nonuniversal positive-definite integral quadratic form with truant t. Show that

$$q' := q \perp \langle t + 1 \rangle \otimes \langle 1, 1, 1, 1 \rangle \perp \langle 2t + 1 \rangle$$

represents all positive integers besides t. (Hint: Use Lagrange's four square theorem). Give explicit examples of a positive-definite integral quadratic forms that represents all integers besides t for $t = 1, 2, 3$.

Problem 8 (Xinling). Do one of the following two problems:

(a) The Hasse–Minkowski principle fails on more general functions. As a simple example, show that

$$f(x) = (x^2 - 2)(x^2 - 17)(x^2 - 34)$$

fails by separately discussing the cases $p = 2$, $p = 17$ and $p \neq 2, 17$.

(b) Prove Theorem 3.2 by a proof similar to 3.1's: let $a, b \in \mathbb{Q}_2^\times$ where $a = 2^k x$ and $b = 2^l y$ with $l, k \in \mathbb{Z}$ and $x, y \in \mathbb{Z}_2^\times$, then

$$(a, b)_2 = (-1)^{\frac{x-1}{2}\frac{y-1}{2} + l\frac{x^2-1}{8} + k\frac{y^2-1}{8}}.$$

Problem 9 (David). Consider the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$, where α has minimal polynomial $g(t) = t^3 - 9t - 9$. It can be shown that $\mathbb{Q}(\alpha)$ is the splitting field for g, so $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension of degree 3. Use Theorem 17 to show that $\mathrm{tr}_K \cong \langle 1, 1, 1 \rangle$.