MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 9

Problem 1. Let k be a field. In this problem, call a relation $<$ on k a *field order* if it is a total order which satisfies $a < b \implies a + c < b + c$ and $0 < a, 0 < b \implies 0 < ab$. Call a subset $P \subseteq \mathsf{k}^\times$ satisfying $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cup (-P) = \mathsf{k}^\times$ a *positive cone*. Show that there is a bijection between field orders on k and positive cones on k taking $<$ to $P_< = \{\lambda \in \mathsf{k}^\times \mid 0 < \lambda\}$.

Problem 2. Recall the Abelian group $(Q(\mathsf{k}), \cdot)$ we introduced on p.31 of the course notes. Define a new operation $\circ$ on $Q(\mathsf{k})$ given by

$$(2\mathbb{Z}, \lambda\mathsf{k}^{\boxtimes}) \circ (2\mathbb{Z}, \mu\mathsf{k}^{\boxtimes}) = (2\mathbb{Z}, \mathsf{k}^{\boxtimes})$$
$$(2\mathbb{Z}, \lambda\mathsf{k}^{\boxtimes}) \circ (1 + 2\mathbb{Z}, \mu\mathsf{k}^{\boxtimes}) = (2\mathbb{Z}, \lambda\mathsf{k}^{\boxtimes})$$
$$(1 + 2\mathbb{Z}, \lambda\mathsf{k}^{\boxtimes}) \circ (1 + 2\mathbb{Z}, \mu\mathsf{k}^{\boxtimes}) = (1 + 2\mathbb{Z}, \lambda\mu\mathsf{k}^{\boxtimes}).$$

(a) Check that $(Q(\mathsf{k}), \cdot, \circ) \cong \mathrm{W}(\mathsf{k})/\mathrm{I}(\mathsf{k})^2$ as commutative rings. (We already proved in Proposition 13.1 that $Q(\mathsf{k}) \cong \mathrm{W}(\mathsf{k})$ as Abelian groups, so your task here is to show that $\circ$ and $\otimes$ are compatible via the isomorphism.)

(b) Show that $Q(\mathsf{k})$ is isomorphic to the ring $\mathbb{Z}[G]/J$ where $G = \mathsf{k}^\times/\mathsf{k}^{\boxtimes}$, $\mathbb{Z}[G]$ is the integral group ring for G, and J is the ideal of $\mathbb{Z}[G]$ generated by

$$[1] + [-1], \quad [1] + [\lambda\mu] - [\lambda] - [\mu]$$

for all $\lambda, \mu \in G$. (Here we are writing $[g]$ for the basis element of $\mathbb{Z}[G]$ corresponding to $g \in G$.) *Bonus*: Show that the same statement holds with $\mathbb{Z}$ replaced by $\mathbb{Z}/4\mathbb{Z}$.

Problem 3. (a) Prove that there is a unique ordering on $\mathbb{Q}$.

(b) Find two distinct orderings on $\mathbb{Q}(\sqrt{2})$. *Bonus*: Show that there are only two orderings on $\mathbb{Q}(\sqrt{2})$. For the remainder of this problem, you may assume this fact.

(c) Find a quadratic form that has signature 0 with respect to one of your orderings, and nonzero signature with respect to the other. Is this class torsion in $\mathrm{W}(\mathbb{Q}(\sqrt{@}))$?

(d) Find a torsion class in $\mathrm{W}(\mathbb{Q}(\sqrt{2}))$ which is not defined over $\mathbb{Q}$.

(e) Is there a nontorsion class in $\mathrm{W}(\mathbb{Q})$ which becomes torsion in $\mathrm{W}(\mathbb{Q}(\sqrt{2}))$?