Homework 8F 4(a)

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[1] *MATH 412*

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Proposition (Problem 4(a)'s tricky direction)**.** *Let λ be an element of $\mathsf{k}^\times$, and let q be a regular quadratic form over k with $\dim q = 2m$. Then, $q \cong \langle \lambda \rangle \otimes q \implies q \cong q_1 \perp \cdots \perp q_m$, where each q_i is a binary form such that $q_i \cong \langle \lambda \rangle \otimes q_i$.*

Proof. We will show this by induction on the dimension (with an induction step size of 2). For the base case when $\dim q = 2$, the equality is trivially established.

Suppose that for $\dim q' = 2(m-1)$, $q' \cong \langle \lambda \rangle \otimes q' \implies q' \cong q_1 \perp \ldots \perp q_{m-1}$ in which $q_i \cong \langle \lambda \rangle \otimes q_i$ are binary forms $\forall i < m-1$. Let's find the case when $\dim q = 2m$. Pick some diagonalization so that $q = \langle \mu_1, \ldots, \mu_{2m} \rangle$ and $\langle \lambda \rangle \otimes q = \langle \lambda\mu_1, \ldots, \lambda\mu_{2m} \rangle$. If two quadratic forms $q \cong \langle \lambda \rangle \otimes q$, then, by Witt's chain equivalence theorem, $\exists$ a binary subform[1] f of q and a binary subform g of $\langle \lambda \rangle \otimes q$ such that $f \cong g$. This follows from the existence of chain equivalence and the definition of simple equivalence.[2] This means that $\exists i, j, k, l \in \mathbb{N}$ such that $\langle \mu_i, \mu_j \rangle \cong \langle \lambda\mu_l, \lambda\mu_k \rangle$. However, we are allowed to permute the diagonal elements to different positions while leaving the form isometric. Hence, WLOG, we can demand that $i = l = 2m-1$ and $j = k = 2m$, *i.e.* $\langle \lambda \rangle \otimes q_m \cong q_m$ in which $q_m = \langle \mu_{2m-1}, \mu_{2m} \rangle$. Then, by the Witt's cancellation theorem, we can cancel out this part and find that $\left\langle \mu_1, \ldots, \mu_{2(m-1)} \right\rangle \cong \left\langle \lambda\mu_1, \ldots, \lambda\mu_{2(m-1)} \right\rangle$. Invoking the induction hypothesis, we can infer that the proposition holds for the $\dim q = 2m$ case. $\square$

Remark. What I learned from this proof is that induction proofs for quadratic forms' equivalence have a "natural step size" of 2 due to the combination of the chain equivalence theorem and the cancellation theorem.

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[1] By a binary subform of q, we mean, quite intuitively, a binary quadratic form that is a part of the orthogonal sum of q.

[2] Note that simple equivalence also permits equivalence between unary subforms, but it that case we can always group two unary subforms into a binary subform.