TOPICS IN ALGEBRA HW 6 WEEK 8, PROBLEM 4(A) SOULTION

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4. Let λ be an element of $\mathsf{k}^\times$ and let q be a regular quadratic form over k with $\dim q = 2m$.

(a): Show that $q \cong \langle\lambda\rangle \otimes q$ if and only if $q \cong q_1 \perp \cdots \perp q_m$, where each q_i is a binary form such that $q_i \cong \langle\lambda\rangle \otimes q_i$.

Proof. The backwards direction is trivial. Assuming $q \cong \sum_i q_i$ where each $q_i = \langle\lambda\rangle \otimes q_i$, we have

$$\langle\lambda\rangle \otimes q \cong \sum_i \langle\lambda\rangle \otimes q_i \cong \sum_i q_i \cong q.$$

The forward direction is harder. It suffices to show that, assuming $q \cong \langle\lambda\rangle \otimes q$, that there exists q_1, a binary form such that $q_1 \cong \langle\lambda\rangle \otimes q_1$, and q', any form, such that $q \cong q_1 \perp q'$. This is because, if we multiply by $\langle\lambda\rangle$, we get

$$q_1 \perp q' \cong q \cong \langle\lambda\rangle \otimes q \cong \langle\lambda\rangle \otimes q_1 \perp \langle\lambda\rangle \otimes q' \cong q_1 \perp \langle\lambda\rangle \otimes q',$$

by cancellation, we get $q' \cong \langle\lambda\rangle \otimes q'$. Then we would be done by induction on the dimension of q because q' will still have even dimension, and the base case $m = 0$ (or $m = 1$) is trivial.

For the case $m = 1$, q itself is binary so the result immediately follows.

If $\lambda \in \mathsf{k}^{\boxtimes}$ the result is immediate, so we assume that λ is not a square in k. Let $n = \dim q = 2m$. Diagonalize q so that we obtain

$$q \cong \langle\alpha, \mu_2, \ldots, \mu_n\rangle \cong \langle\lambda\rangle \otimes \langle\alpha, \mu_2, \ldots, \mu_n\rangle \cong \langle\lambda\alpha, \lambda\mu_2, \ldots, \lambda\mu_n\rangle$$

for some α, μ_i all nonzero (q is regular). Then, we know that $\lambda\alpha \in D(q)$ because $\lambda\alpha$ appears in the rightmost diagonalization above (evaluate

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at $(1, 0, \ldots, 0)$). Thus, considering the leftmost diagonalization above, there exists $x, y_2, \ldots, y_n \in \mathsf{k}$ such that

$$\alpha x^2 + \sum_{i=2}^{n} \mu_i y_i^2 = \lambda\alpha.$$

Rearranging, this tells us that

$$\alpha(\lambda - x^2) = \sum_{i=2}^{n} \mu_i y_i^2,$$

which implies that $\alpha(\lambda - x^2) \in D(\langle \mu_2, \ldots, \mu_n \rangle)$. Then by the representation criterion, we have that

$$q \cong \langle \alpha \rangle \perp \langle \mu_2, \ldots, \mu_n \rangle \cong \langle \alpha, \alpha(\lambda - x^2) \rangle \perp q'$$

for some q'.

We claim that

$$\langle \alpha, \alpha(\lambda - x^2) \rangle \cong \langle \lambda \rangle \otimes \langle \alpha, \alpha(\lambda - x^2) \rangle \cong \langle \lambda\alpha, \lambda\alpha(\lambda - x^2) \rangle.$$

It is obvious, and one can easily check, that the two forms $\langle \alpha, \alpha(\lambda - x^2) \rangle$ and $\langle \lambda\alpha, \lambda\alpha(\lambda - x^2) \rangle$ have the same determinant up to squares. Furthermore, they both represent $\lambda\alpha$. The first one, $\langle \alpha, \alpha(\lambda - x^2) \rangle$, represents $\lambda\alpha$ when evaluated at $(x, 1)$. The second one, $\langle \lambda\alpha, \lambda\alpha(\lambda - x^2) \rangle$, represents $\lambda\alpha$ when evaluated at $(1, 0)$. Lastly, the two forms are obviously binary, and are regular because $\alpha \neq 0$ and $\lambda - x^2 \neq 0$ because we assumed λ is not a square in k. Thus, $\langle \alpha, \alpha(\lambda - x^2) \rangle$ and $\langle \lambda\alpha, \lambda\alpha(\lambda - x^2) \rangle$ are regular binary forms who have the same determinant up to squares and represent a common element of k. This shows that indeed

$$\langle \alpha, \alpha(\lambda - x^2) \rangle \cong \langle \lambda \rangle \otimes \langle \alpha, \alpha(\lambda - x^2) \rangle \cong \langle \lambda\alpha, \lambda\alpha(\lambda - x^2) \rangle.$$

Then, letting $q_1 = \langle \alpha, \alpha(\lambda - x^2) \rangle$, we can write $q = q_1 \perp q'$ where $q_1 \cong \langle \lambda \rangle \otimes q_1$, which we said was sufficient.

(Do note that x could be 0, but this does not affect our argument). $\square$
