MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 8

Problem 1. Suppose that $\langle \lambda_1, \dots, \lambda_n \rangle \cong \langle \mu_1, \dots, \mu_n \rangle$ over k with $\lambda_i, \mu_i \in \mathsf{k}^\times$. Prove that
$$\prod_{i=1}^{n} (\langle \lambda_i \rangle - 1) = \prod_{i=1}^{n} (\langle \mu_i \rangle - 1)$$
in $\mathrm{GW}(\mathsf{k})$.

Problem 2. Show that if $\mathrm{I}(\mathsf{k})^2 = 0$, then every regular binary form over k is universal.

Problem 3. Show that the cardinality of $\mathrm{W}(\mathsf{k})$ is finite if and only if -1 is a sum of squares in k and $\mathsf{k}^\times / \mathsf{k}^{\boxtimes}$ is finite. (*Hint*: ◯.)

Problem 4. Let λ be an element of $\mathsf{k}^\times$ and let q be a regular quadratic form over k with $\dim q = 2m$.

(a) Show that $q \cong \langle \lambda \rangle \otimes q$ if and only if $q \cong q_1 \perp \cdots \perp q_m$, where each q_i is a binary form such that $q_i \cong \langle \lambda \rangle \otimes q_i$.
(b) Suppose that $q \cong \langle -1 \rangle \otimes q$ and deduce that $q \otimes q$ is hyperbolic.
(c) Conclude that $2q = 0 \in \mathrm{W}(\mathsf{k})$ implies that $q^2 = 0 \in \mathrm{W}(\mathsf{k})$.

Problem 5. Compute the image of $q = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n$ in $\mathrm{GW}(\mathsf{k})$ and $\mathrm{W}(\mathsf{k})$ for $\mathsf{k} = \mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}_p$, $p > 2$ prime.