MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 6

Problem 1. Prove or learn the proofs of Propositions 10.5, 10.6, and 10.7 from the course notes. Write up one of the proofs in your own words.

Problem 2. Prove that the tensor product of two regular quadratic forms is regular. (This was asserted but not proved in class.)

Problem 3. The *Witt index* $i(V)$ of a symmetric bilinear form (V, B) is half the dimension of the hyperbolic part of the form in its Witt decomposition, *i.e.* $\frac{1}{2}\dim V_h$ when $V \cong V_t \perp V_h \perp V_a$ with V_t totally isotropic, V_h hyperbolic, and V_a anisotropic. Prove the following statements for all regular symmetric bilinear forms (V, B), (W, B'):
(a) $i(V \otimes W) \geq i(V) \cdot \dim W$ and
(b) $i(V \perp W) \leq i(V) + \dim W$.

Problem 4. Let f be a regular group form. Show that for any regular form g,

$$D(f) \cdot D(f \otimes g) = D(f \otimes g).$$

[Added Oct. 15: This appears to either be false or highly subtle in general! There is a straightforward proof when the base field is Pythagorean (sums of squares are sqaures). Can you find a counterexample over a non-Pythagorean field?]

Problem 5 (Trace forms). A k-*algebra* is a k-vector space A which is also a commutative ring for which the multiplication map $A \times A \to A$ is k-bilinear. In particular, for $a \in A$, the multiplication-by-a map $m_a : A \to A$ is k-linear. Recall that any linear endomorphism of a finite-dimensional vector space has a well-defined *trace* in k. If $\dim_{\mathsf{k}} A < \infty$, define $\mathrm{tr}_{A/\mathsf{k}} : A \to \mathsf{k}$ to be the map taking a to the trace of m_a.

(a) If A is a finite-dimensional k-algebra, prove that the assignment $(a, b) \mapsto \mathrm{tr}_{A/\mathsf{k}}(ab)$ is a symmetric bilinear form on A. (This is called the *trace form* of A and will be denoted $(A, \mathrm{tr}_{A/\mathsf{k}})$ [slightly overloading the notation $\mathrm{tr}_{A/\mathsf{k}}$].)
(b) If B is another finite-dimensional k-algebra, check that $A \times B$ is again a finite-dimensional k-algebra when given the coordinate-wise multiplication. Then prove that

$$(A \times B, \mathrm{tr}_{A \times B/\mathsf{k}}) \cong (A, \mathrm{tr}_A) \perp (B, \mathrm{tr}_B).$$

(c) Show that $A \otimes B$ is a k-algebra with product induced by $(a \otimes b) \cdot (a' \otimes b') = (ab) \otimes (bb')$. Then prove that

$$(A \otimes B, \mathrm{tr}_{A \otimes B/\mathsf{k}}) \cong (A, \mathrm{tr}_A) \otimes (B, \mathrm{tr}_B).$$