MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 5

Problem 1 (Another chance at Week 4's Problem 5). Show that a binary form $\langle 1, -a\rangle$ over $\mathbb{Q}$ is universal if and only if $a \in \mathbb{Q}^{\boxtimes}$. *Hint*: First show (by the First Representation Criterion) that the claim holds over any field k when the universality hypothesis is replaced with isotropicity. Now show that a regular form is isotropic over $\mathbb{Q}$ if and only if it is universal. (Don't be afraid of prime numbers.)

Problem 2. Let k be a field and suppose $F_i \subseteq \mathsf{k}$ is a subfield for $i \in I$. Define $F = \bigcap_{i\in I} F_i$. Show that the natural homomorphism $F^\times/F^{\boxtimes} \to \prod_{i\in I} F_i^\times/F_i^{\boxtimes}$ is an injection. Conclude that if $|I| < \infty$ and $[F_i^\times : F_i^{\boxtimes}] < \infty$ for all $i \in I$, then $[F^\times : F^{\boxtimes}] < \infty$.

Problem 3. In a hyperbolic space, a maximal totally isotropic subspace is called a *Lagrangian*. Show that every hyperbolic space is the orthogonal sum of two Lagrangians. Draw a picture of such a Lagrangian decomposition of $h = \langle 1, -1\rangle$ over $\mathbb{R}$.

Problem 4. Use Witt's Decomposition Theorem to prove *Sylvester's Law of Inertia*:

> Fix $n \geq 1$. For every n-ary regular quadratic form q over $\mathbb{R}$, there are unique integers n_+ and n_- such that $q \cong n_+ \langle 1\rangle \perp n_- \langle -1\rangle$.

In particular, the equivalence class of every regular quadratic form over $\mathbb{R}$ is determined by its dimension ($n = n_+ + n_-$) and signature ($n_+ - n_-$).

Problem 5. Exhibit a chain equivalence between $\langle 2, 3, 6\rangle$ and $\langle 1, 1, 1\rangle$ over $\mathbb{Q}$. *Bonus*: Show that the equation $2x^2 + 3y^2 + 6z^2 = 7$ cannot be solved over $\mathbb{Q}$.