MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 4

Problem 1. Prove Lemma 6.6 from the course notes, being careful not to use any of the results that depend upon it.

Problem 2. For a form $\varphi = \sigma \perp \tau$, show that

$$D(\varphi) = \bigcup_{s \in D(\sigma), t \in D(\tau)} D(\langle s, t \rangle).$$

From this, deduce that

$$D(\langle \lambda \rangle \perp \tau) = \bigcup_{t \in D(\tau)} D(\langle \lambda, t \rangle).$$

Problem 3. Prove that $\langle 2, 3 \rangle \cong \langle 1, 6 \rangle$ over $\mathbb{R}$, and over $\mathbb{F}_p$ where $p > 3$ is prime, but that these forms are not equivalent over $\mathbb{F}_3$.

Problem 4. Let $A = (a_{ij})$ be a symmetric 3×3 matrix over k. Set

$$d_1 = a_{11}, \quad d_2 = a_{11}a_{22} - a_{12}^2, \quad d_3 = \det A,$$

and assume that $d_1 d_2 d_3 \neq 0$. Prove that

$$A \cong \langle d_1, d_1 d_2, d_2 d_3 \rangle \cong \langle d_1, d_2/d_1, d_3/d_2 \rangle .$$

Bonus: Generalize this diagonalization method to $A \in \mathrm{Sym}_{n \times n}(\mathsf{k})$.

Problem 5. Show that a binary form $\langle 1, -a \rangle$ over $\mathbb{Q}$ is universal if and only if $a \in \mathbb{Q}^{\boxtimes}$.