MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 3

Problem 1. Let (V, q) be a quadratic space with associated symmetric bilinear form B. Prove that the following statements hold for all $x, y \in V$, and furthermore interpret each label.

(a) If $B(x, y) = 0$, then $q(x + y) = q(x) + q(y)$ [Pythagorean theorem for a right triangle].
(b) $q(x + y) + q(x - y) = 2q(x) + 2q(y)$ [Pythagorean theorem for a parallelogram].
(c) If $q(x) = q(y)$, then $B(x + y, x - y) = 0$ [orthogonality of diagonals of a rhombus].

Problem 2. Let V be a k-vector space and let $\beta : V \times V \to \mathsf{k}$ be a (not necessarily symmetric) bilinear form. Prove that $\beta = \sigma + \tau$ where σ is a symmetric bilinear form and τ is a skew-symmetric bilinear form. (Skew-symmetry says that $\tau(y, x) = -\tau(x, y)$ for all $x, y \in V$.)

Problem 3. For a k-vector space V, let $\operatorname{Bil} V$ denote the set of *all* (not necessarily symmetric) bilinear forms $\beta : V \times V \to \mathsf{k}$.

(a) Generalize the "right vertical side of the quadratic square" to show that $\operatorname{Bil} V$ is in bijective correspondence with $M_{n\times n}(\mathsf{k})$.
(b) Let A be the matrix associated with $\beta \in \operatorname{Bil} V$. Show that β is symmetric if and only if A is symmetric, and show that β is skew-symmetric if and only if $A = -A^\top$ (recalling that $\operatorname{char} \mathsf{k} \neq 2$).

Problem 4. Prove that the quadratic form $ax^2 - ay^2$ is equivalent to $x^2 - y^2$ for all $a \in \mathsf{k}^\times$.

Problem 5. Let (V, B) be a symmetric bilinear form with associated quadratic space (V, q). Prove that (V, B) is regular if and only if the following condition holds:

If $y \in V$ and $q(x + y) = q(x) + q(y)$ for all $x \in V$, then $y = 0$.

Problem 6. For $a, b \in \mathsf{k}^\times$, show that $b \in D(x^2 + ay^2)$ if and only if $bx^2 + aby^2$ is equivalent to $x^2 + ay^2$.