MATH 412: TOPICS IN ALGEBRA
HOMEWORK DUE FRIDAY WEEK 2

Remark 1. Make sure to review the "Homework" portion of the syllabus before writing up your solutions!

Problem 1. Suppose that k is a field and $H \leq \mathsf{k}^\times$ is a finite subgroup of the multiplicative group $\mathsf{k}^\times = \mathsf{k} \smallsetminus \{0\}$. Prove that H is cyclic.

Problem 2. Suppose that U, V, and W are k-vector spaces. Prove that

$$\mathrm{Hom}_{\mathsf{k}}(U \oplus V, W) \cong \mathrm{Hom}_{\mathsf{k}}(U, W) \oplus \mathrm{Hom}_{\mathsf{k}}(V, W)$$

as vector spaces by constructing a natural isomorphism. What can you conclude about $(U \oplus V)^*$ versus $U^* \oplus V^*$?

Problem 3. Let V and W be k-vector spaces.

(a) Prove that $\mathrm{Hom}_{\mathsf{k}}(\mathsf{k}, V) \cong V$.

(b) Suppose additionally that V and W are finite-dimensional. Determine the dimensino of $\mathrm{Hom}_{\mathsf{k}}(V, W)$ in terms of $\dim V$ and $\dim W$.

Problem 4. Let V be a vector space with ordered basis $v_1, \ldots, v_n$, let W be a vector space with ordered basis $w_1, \ldots, w_m$, and suppose $f \colon V \to W$ is a linear transformation with matrix A with respect to these bases. Show that the dual transformation $f^* \colon W^* \to V^*$ has matrix $A^\top$ with respect to the ordered dual bases $w_1^*, \ldots, w_m^*$ and $v_1^*, \ldots, v_n^*$.

Problem 5. Suppose that V and W are k-vector spaces. Prove that the linear transformation

$$\mathrm{Hom}_{\mathsf{k}}(V, W) \xrightarrow{(\,)^*} \mathrm{Hom}_{\mathsf{k}}(W^*, V^*)$$

is injective. Use Problem 3(b) to conclude that this map is an isomorphism $\mathrm{Hom}_{\mathsf{k}}(V, W) \cong \mathrm{Hom}_{\mathsf{k}}(W^*, V^*)$ when V and W are finite-dimensional.

Problem 6. Observe that the square of any integer is congruent to 0 or 1 modulo 4.

(a) Use the above fact to deduce that if a is a sum of two squares of integers, then $a \not\equiv 3 \pmod 4$.

(b) Use congruences modulo 4 and 3 to deduce that there are no integer solutions to the equation $x^2 + y^2 = 21$.

(A theorem from elementary number theory says that a positive integer is a sum of two squares if and only if its prime decomposition contains no prime congruent to 3 modulo 4 raised to an odd power. You may not invoke this theorem in your proof of (b)!)