MATH 342: TOPOLOGY
UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS
ON COMPACT METRIC SPACES

Fix metric spaces (X, d_X) and (Y, d_Y). Recall that $f\colon X \to Y$ is continuous iff for every $\varepsilon > 0$ and every $x \in X$, there exists $\delta = \delta_x > 0$ such that

$$x \in B(x, \delta) \implies f(x) \in B(f(x), \varepsilon).$$

In other words, $fB(x, \delta_x) \subseteq B(f(x), \varepsilon)$. We call f *uniformly continuous* when for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Crucially, the δ in the definition of uniform continuity depends only on ε and not on x_0, x_1. The reader should check that uniform continuity implies continuity, but the converse does not hold in general (think about $x \mapsto 1/x$ as a function $(0, \infty) \to \mathbb{R}$). Nevertheless, the converse does hold when X is compact:

Theorem 1. *Suppose (X, d_X), (Y, d_Y) are metric spaces and that X is compact. If $f\colon X \to Y$ is continuous, then f is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. Since f is continuous, for each $x \in X$ there exists $\delta_x > 0$ such that $fB(x, \delta_x) \subseteq B(f(x), \varepsilon/2)$. The set

$$\mathscr{U} := \{B(x, \delta_x/2) \mid x \in X\}$$

forms an open cover of X, which has a finite subcover

$$\mathscr{U}' = \{B(x_1, \delta_{x_1}/2), \ldots, B(x_n, \delta_{x_n}/2)\}$$

since X is compact.

Set $\delta = \min\{\delta_{x_1}/2, \ldots, \delta_{x_n}/2\}$. Fix $x, y \in X$ such that $d_X(x, y) < \delta$. We aim to show that $d_Y(f(x), f(y)) < \varepsilon$. Since $\mathscr{U}'$ is an open cover of X, there exist x_i such that $x \in B(x_i, \delta_{x_i}/2)$. Additionally,

$$d_X(x_i, y) \le d(x_i, x) + d(x, y) < \delta_{x_i}/2 + \delta \le \delta_{x_i}$$

so $x, y \in B(x_i, \delta_{x_i})$. It follows that $d_Y(f(x_i), f(x)) < \varepsilon/2$ and $d_Y(f(x_i), f(y)) < \varepsilon/2$. By the triangle inequality,

$$d_Y(f(x), f(x)) \le d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that f is uniformly continuous. $\square$