# MATH 342: TOPOLOGY
# FUNCTOR EXAMPLES

As a supplement to our functor lecture, consider the following examples.

**Example 1.** Fix a field $\mathsf{k}$. There is a free functor $F\colon \mathsf{Set} \to \mathsf{Vect}_\mathsf{k}$ taking a set $S$ to

$$FS = \bigoplus_S F.$$

Here $\bigoplus_S F$ is the vector space of functions $f\colon S \to F$ with $f(s) \neq 0$ for only finitely many $s \in S$. The linear structure is given by pointwise addition and scaling.

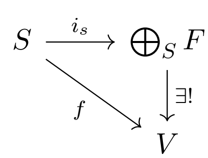
Note that a basis for $\bigoplus_S F$ is given by $\{\chi_s \mid s \in S\}$, the set of characteristic functions of singleton subsets of $S$. As such, every linear map out of $\bigoplus_S F$ is specified by its value on $\chi_s$, $s \in S$, extended linearly.

The above observation allows us to define $Ff$ for $f\colon S \to T$ a function. Indeed, we define $(Ff)(\chi_s) := \chi_{f(s)}$. It is clear that $F\,\mathrm{id}_S = \mathrm{id}_{FS}$, and for composable functions $S \xrightarrow{f} T \xrightarrow{g} U$, we have

$$(Fgf)(\chi_s) = \chi_{g(f(s))} = (Fg)(\chi_{f(s)}) = (Fg)(Ff)(\chi_s).$$

This shows that $F$ is a functor.

Note that $FS = \bigoplus_S F$ satisfies a universal property. Consider the function $i_S\colon S \to \bigoplus_S F$ given by $i_S(s) = \chi_s$. Then for $V$ any $\mathsf{k}$-vector space and $f$ any function $f\colon S \to V$, there exists a unique linear transformation $\bigoplus_S F \to V$ making the diagram




commute. Indeed, $\bigoplus_S F \to V$ is given by $\chi_s \mapsto f(s)$ (extended linearly).

It is in the fact the case that the forgetful functor $U\colon \mathsf{Vect}_\mathsf{k} \to \mathsf{Set}$ and $F$ fit into the so-called *free-forgetful adjunction*. This says that

$$\mathsf{Vect}_\mathsf{k}(FS, V) \cong \mathsf{Set}(S, UV)$$

naturally. (The 'natural' part is a technical condition saying that we are actually dealing with a natural isomorphism of functors, but I won't unpack that here.) This essentially says that linear maps out of $FS$ are in bijection with set maps out of $S$, and is another interpretation of the universal property of $FS$.

**Example 2.** The `#remote` Slack channel thought about the following example. Let $\mathsf{Op}_*$ denote the category with objects $(U, x)$ where $U$ is an open subset of some Euclidean space $\mathbb{R}^k$ and $x \in U$, and morphisms $f\colon (U, x) \to (V, y)$ given by differentiable functions $f\colon U \to V$ such that $f(x) = y$. I claim that the derivative is a functor $D\colon \mathsf{Op}_* \to \mathsf{Vect}_\mathbb{R}$ defined as follows:

Given an object $(U, x)$ of $\mathsf{Op}_*$ where $U$ is an open subset of $\mathbb{R}^n$, define $D(U, x)$ to be $\mathbb{R}^k$. If $f\colon (U, x) \to (V, y)$ is a morphism in $\mathsf{Op}_*$ and $V \subseteq \mathbb{R}^m$, define $Df$ to be $D_x f\colon \mathbb{R}^n \to \mathbb{R}^m$ to be the derivative of $f$ at $x$ (considered as a linear transformation).

We now check functoriality. Since the identity map is linear, it is clear that $D_x\,\mathrm{id}_{U,x} = \mathrm{id}_{\mathbb{R}^m}$. Now suppose that $(U, x) \xrightarrow{f} (V, y) \xrightarrow{g} (W, w)$ are composable morphisms in $\mathsf{Op}_*$. Then by the

**multivariable chain rule**,

$$D_x(gf) = D_y(g)D_x(f).$$

The upshot is that the chain rule is essential in identifying the multivariable derivative as a functorial construction!

Now some readers may prefer to think in coordinates and replace $D_x f$ with the Jacobian matrix $Jf(x)$ of $f$ at $x$. This is the $m \times n$ matrix

$$\left( \frac{\partial f_i}{\partial x_j}(x) \right)_{ij}$$

of partial derivatives of $f$ at $x$. We may consider $J$ as a functor $\mathsf{Op}_* \to \mathsf{Mat}$ taking $(U \subseteq \mathbb{R}^n, x)$ to $n$ and $f\colon (U,x) \to (V,y)$ to the Jacobian matrix $Jf(x)$. The chain rule again verifies that this is a functor.