# Day 33

## Learning Goals

- Seifert-van Kampen for $\Pi_1$
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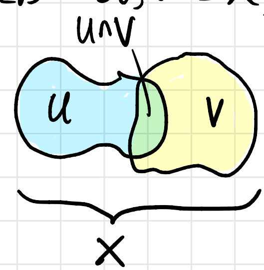
- Pushouts in Gp.
- Examples

Recall pushout in Top:
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ C \arrow[r] & B \amalg C / f(a) \sim g(a) \end{tikzcd}.$$

If $X = U \cup V$ for open sets $U, V \subseteq X$, then

$$\circledast \quad \begin{tikzcd} U \cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}.$$




**Thm** [SvK for $\Pi_1$] Suppose $X = U \cup V$ for open sets $U, V$. Then applying $\Pi_1$ to $\circledast$ yields a pushout diagram in Gpd, the category of groupoids:

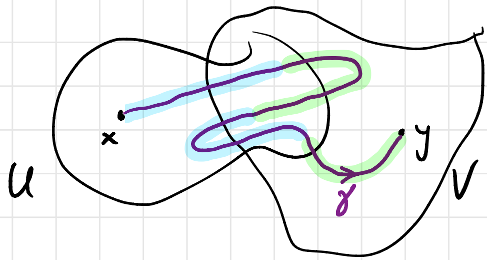
$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Pi_1 U \arrow[d] \\ \Pi_1 V \arrow[r] & \Pi_1 X \end{tikzcd}.$$

Pf Idea Given a commutative diagram of groupoids

$$(\dagger)\quad \begin{tikzcd} \Pi_1(U\cap V) \arrow[r] \arrow[d] & \Pi_1 U \arrow[d] \arrow[ddr, bend left, "f"] & \\ \Pi_1 V \arrow[r] \arrow[drr, bend right, "g"'] & \Pi_1 X \arrow[dr, dashed, "\Phi"] & \\ & & G \end{tikzcd}$$

define
$$\Phi x = \begin{cases} f(x) & \text{if } x\in U \\ g(x) & \text{if } x\in V \end{cases}$$
on objects

and given $[\gamma]\in \Pi_1 X(x,y)$ do the following:



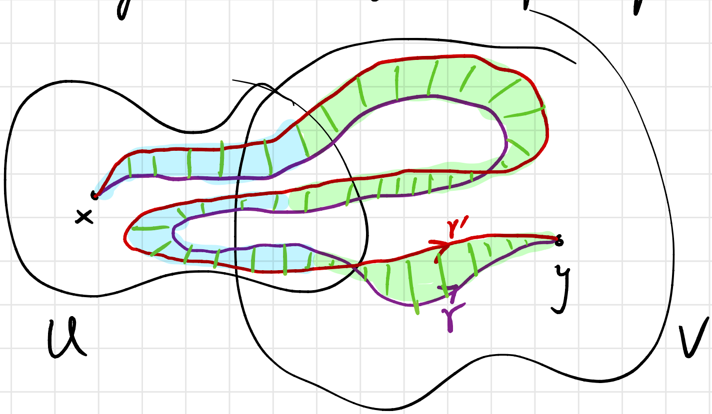

I.e. use compactness of $I$ to write $[\gamma]$ as a concatenation of paths $[\gamma_n \cdots \gamma_1]$ with $\gamma_i(I)\subseteq U$ or $V$ for all $i$. Let $f_i = \begin{cases} f & \text{if } \gamma_i(I)\subseteq U \\ g & \text{if } \gamma_i(I)\subseteq V \end{cases}$ and then

define $\Phi[\gamma] := f_n[\gamma_n]\cdot f_{n-1}[\gamma_{n-1}]\cdots f_1[\gamma_1]$.

It remains to show that $\Phi$ is well-defined and is the unique functor making the diagram commute.

For well-def'n, suppose $\gamma'\simeq\gamma$ and choose a htpy $h: \gamma' \Rightarrow \gamma$. Use compactness to subdivide $I\times I$

into rectangles with images completely in $U$ or $V$:




This supplies homotopies b/w $\gamma_i'$, $\gamma_i$ for each $i$ whence $f_i[\gamma_i'] = f_i[\gamma_i] \Longrightarrow$ well-defined.

$\Phi$ makes the diagram commute b/c $f, g$ agree on pts + paths in $U \cap V$. The form of $\Phi$ is mandated by the commutativity of the outer triangles in (†). $\square$

**Thm** [SvK for $\pi_1$] Suppose $X = U \cup V$ for $U, V \subseteq X$ open and suppose $x_0 \in U \cap V$. If $U \cap V$ is path connected, then

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \pi_1(U, x_0) \arrow[d] \\
\pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd}$$

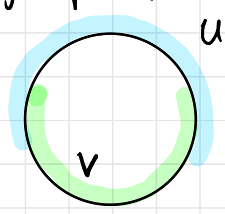
is a pushout in Gp.

**Pf Idea** First replace $U, V, X$ with $U', V', X'$, the path components of $x_0$ in each space. These have the same $\pi_1(-, x_0)$. The diagram (†) restricts to a pushout the connected component of $x_0$ and — since $U \cap V$ is path connected — the inclusions

$$\pi_1(U \cap V, x_0) \subseteq \Pi_1(U \cap V)$$
$$\pi_1(U', x_0) \subseteq \Pi_1 U'$$
$$\pi_1(V', x_0) \subseteq \Pi_1 V'$$
$$\pi_1(X', x_0) \subseteq \Pi_1 X'$$

are equivalences of categories. By abstract nonsense, the corresponding $\pi_1(-, x_0)$ diagram is a pushout in Gp. $\square$

**Note** The theorem (and proof) fail for $U \cap V$ not path conn'd. Consider



## Pushouts in Gp

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & G \arrow[d] \\ H \arrow[r] & \underbrace{G*H/N} \end{tikzcd}$$

amalgamated free product

for $K, G, H$ groups, $f, g$ homomorphisms,

$G*H$ = free product of $G, H$
= coproduct in Gp,

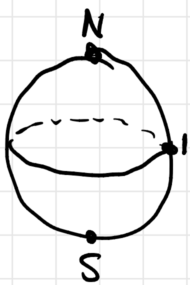
$N$ = normal subgp of $G*H$ gen'd by rel'ns
$f(k) = g(k) \quad \forall k \in K$
i.e. by $f(k)g(k)^{-1} \; \forall k \in K$.

## Examples

① $X = S^2$, $\quad U = S^2 \setminus N \cong D^2 \simeq *$

north pole    south pole

$V = S^2 \setminus S \cong D^2 \simeq *$



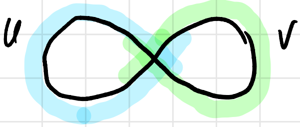

$U \cap V = S^2 \setminus \{N, S\} \cong D^2 \setminus 0 \simeq S^1$

By SvK,

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & e \arrow[d] \\ e \arrow[r] & \pi_1(S^2, 1) \cong e*e/\ldots = e \end{tikzcd}$$

(Think about the geometric meaning of this. Can you give a geometric proof?)

② $X = S^1 \vee S^1$




$U \simeq V \simeq S^1$

$U \cap V = \times \simeq *$

By SvK,

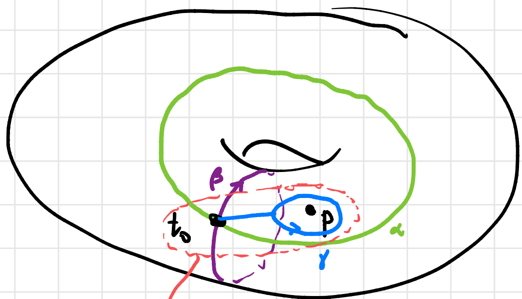
$$\begin{tikzcd} e \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \end{tikzcd}$$

This is the free group on two generators.

③ $X = S^1 \times S^1$

$V = X \setminus \{p\}$
$\simeq S^1 \vee S^1$

$U \cap V = \text{disk} \setminus \{p\}$
$\simeq S^1$.




U = small open disk containing p, $t_0$

By SvK,

$$\begin{tikzcd} \gamma \arrow[d, mapsto] & \mathbb{Z}\{\gamma\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & e \arrow[d] \\ \alpha\beta\alpha^{-1}\beta^{-1} & \mathbb{Z}\{\alpha\} * \mathbb{Z}\{\beta\} \arrow[r] & \pi_1(X, t_0) \end{tikzcd}$$

$$\Longrightarrow \pi_1(X, t_0) \cong F(\alpha, \beta) / \underbrace{\langle \alpha\beta\alpha^{-1}\beta^{-1} \rangle}_{\alpha\beta = \beta\alpha} \cong \mathbb{Z} \times \mathbb{Z}$$