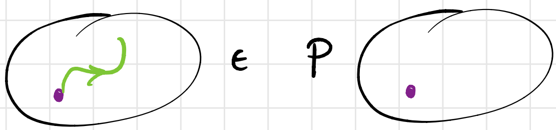
Day 32

Learning Goals

Defn For $X = (X, x_0)$ based, the **based path space** of X is $PX := \mathrm{Top}_*((I,0),(X,x_0))$ with compact-open topology.



Give PX base point const_{x_0}

Define $p : PX \longrightarrow X$, $\gamma \longmapsto \gamma(1)$. Then $p^{-1}\{x_0\} = \Omega X$.

This map is based ($\mathrm{const}_{x_0} \mapsto x_0$) and a fibration:

$z \longmapsto g(z) = $ path in X ending at $h(z,0)$ $=$ starting pt of $h(z,-)$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i"'] & PX \arrow[d, "p"] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & X \end{tikzcd}$$

Define $\tilde{h}(z,t) :=$ path in X with param'n

$$s \longmapsto \begin{cases} g(z)(s(1+t)) & \text{if } 0 \le s \le \frac{1}{1+t} \\ h(z, s(1+t)-1) & \text{if } \frac{1}{1+t} \le s \le 1 \end{cases}$$

Then $\tilde{h}$ is a reparametrization of $h(z,-)\cdot g(z)$

and (check) : · $\tilde{h}$ is based

· $\tilde{h}$ lifts h.

Prop $PX \simeq *$

Pf

$$\begin{tikzcd} * \arrow[r] \arrow[rr, bend right, "id_*"'] & PX \arrow[r] & * \end{tikzcd} \quad ✓$$

$$\begin{tikzcd} PX \arrow[r] \arrow[rr, bend right, "\gamma \mapsto const_{x_0}"'] & * \arrow[r] & PX \end{tikzcd}$$

Define $h : PX \times I \to PX$

$$(\gamma, t) \mapsto \begin{array}{c} s \\ \downarrow \\ \gamma(s+(1-s)t) \end{array}$$

Then $h(-,0) = id_{PX}$,

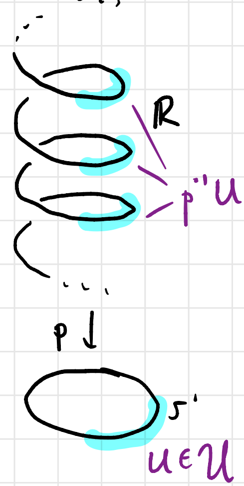
$h(-,1) = const_{\gamma(1)}$,

$h(const_{x_0}, t) = const_{x_0}$. $\square$

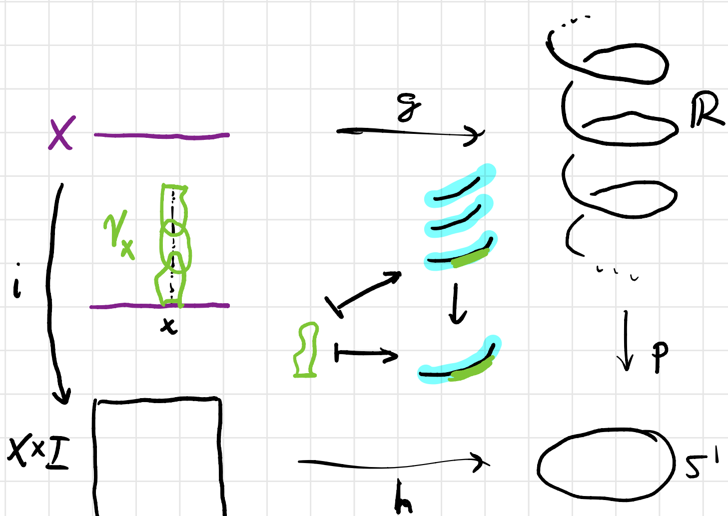
E.g. The map $p : \mathbb{R} \longrightarrow S^1$, $t \longmapsto e^{2\pi i t}$ is a fibration with fiber $\mathbb{Z}$.

Consider
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & \mathbb{R} \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] & S^1 \end{tikzcd}$$

Take an open cover $\mathcal{U}$ of S^1 s.t. for all $U \in \mathcal{U}$,
$p^{-1}U = \{\text{disjoint opens in } \mathbb{R}, \text{ each} \cong U\}$.



Since h is cts, $h^{-1}\mathcal{U}$ is an open cover of $X \times I$. For each $x \in X$ choose finite subcover $V_x \subseteq h^{-1}\mathcal{U}$ of $\{x\} \times I$.



Inductively patch together lift $\tilde{h}$ on each V_x

Works on overlaps b/c p is a local homeomorphism.

Thus get g_x lifting h on $U \mathcal{V}_x \cup (X \times \{0\})$.

Now check — again with the fact that p is a local homeomorphism — that g_x, g_y agree on overlap. Thus

$$\tilde{h}(x,t) = g_x(x,t)$$

is well-defined and lifts h. $\square$

Goal

$$\begin{tikzcd} \Omega S^1 \arrow[r, dashed, "\simeq"'] \arrow[d] & \mathbb{Z} \arrow[d] \\ PS^1 \arrow[r, dashed, "\simeq"'] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, equal] & S^1 \end{tikzcd}$$

If we succeed, $\Omega S^1 \simeq \mathbb{Z} \Rightarrow$

as groups!

$$\pi_1 S^1 \cong \pi_0 \Omega S^1 \cong \pi_0 \mathbb{Z} \cong \mathbb{Z}.$$

Thm Suppose p, q fib'ns and

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

commutes.

If f is a htpy equiv, then f induces a htpy equiv b/w fibers.

Pf The idea is to produce a htpy inverse to f that preserves fibers. Then f will restrict to $p^{-1}b \simeq q^{-1}b$.

By assumption, f has a htpy inverse f' hence $\exists\, h' : ff' \Rightarrow id_D$.

$$\begin{tikzcd}
D \arrow[r, "f'"] \arrow[d] & & E \arrow[d, "p"] \\
D\times I \arrow[r, "h'"'] \arrow[urr, dashed, "\tilde{h}"] \arrow[rr, bend right, "h = qh'"'] & D \arrow[r, "q"'] & B
\end{tikzcd}$$

Check that $g := \tilde{h}(-,1)$ preserves fibers and is a htpy inverse to f.

(Omitting some details regarding homotopies preserving fibers.) $\square$

Cor $\pi_n S^1 \cong \begin{cases} \text{trivial} & n \neq 1 \\ \mathbb{Z} & n = 1 \end{cases}$

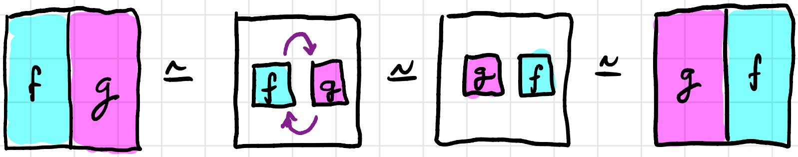
Pf S^1 is conn'd and we already argued $\pi_1 S^1 \cong \mathbb{Z}$. For $n \geq 2$, $\pi_n S^1 \cong \pi_{n-1} \Omega S^1 \cong \pi_{n-1} \mathbb{Z}$

$= [S^{n-1}, \mathbb{Z}]$.

Since S^{n-1} is conn'd for $n>1$, all based maps to $\mathbb{Z}$ are trivial. □

Aside: Thm If $n \geq 2$, then $\pi_n X$ is Abelian.

Sketch for $n=2$



□

π_1 arlor Tricks (corollaries of $\pi_1 S^1 \cong \mathbb{Z}$ — see 6.6.3)

- Brouwer's fixed pt thm for D^2
- Perron-Frobenius: every 3×3 matrix with positive entries has a positive eigenvalue
- Fundamental Thm of Algebra: every nonconstant complex polynomial has a root in $\mathbb{C}$.