Day 31

Learning goals

- Define (co)fibrations via lifting/extension problems
- Mapping path space and mapping cylinder

Everything in the category Top today.

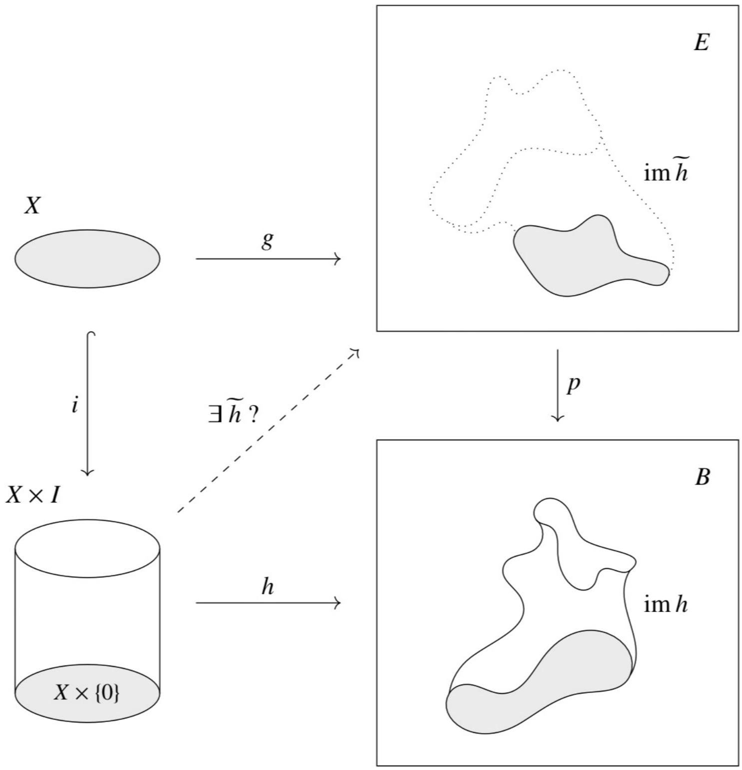
Write $i: X \to X \times I$ for $x \mapsto (x, 0)$.

Defn A map $p: E \to B$ is a fibration when $\forall g, h$

s.t.
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] & B \end{tikzcd}$$
commutes, $\exists \tilde{h}: X \times I \to E$

s.t.
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, "\tilde{h}"] & B \end{tikzcd}$$
commutes.

We say that $\tilde{h}$ lifts the homotopy h.



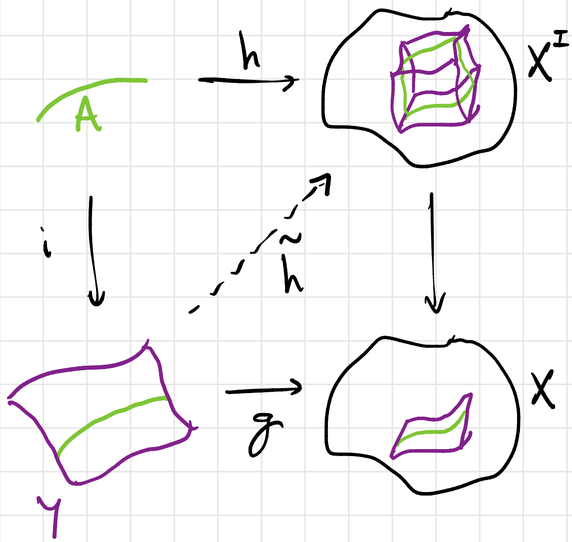
Defn A map $i: A \to Y$ is a cofibration when $\forall\, h, g$ making

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & X^I \arrow[d, "e"] & \gamma \arrow[d, mapsto] \\ Y \arrow[r, "g"'] & X & \gamma(0) \end{tikzcd}$$

commute, $\exists\, \tilde{h}: Y \to X^I$ s.t.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & X^I \arrow[d, "e"] \\ Y \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{h}"] & X \end{tikzcd}$$

commutes. We say that $\tilde{h}$ *extends* the homotopy h.



Slogans
- Fibrations $E \to B$ are maps s.t. if a homotopy in B lifts to E at time 0 then it lifts completely.
- Cofibrations $A \to Y$ are maps s.t. if a homotopy from A extends to Y at time 0 then it extends completely.

Thm Every cts $f: X \to Y$ factors as

$$\begin{tikzcd} X \arrow[r, "g", "\simeq"'] \arrow[dr, "f"'] & E \arrow[d, "p", "\text{fib}"'] \\ & Y \end{tikzcd}$$

$f = pg$

and

$$\begin{tikzcd} X \arrow[d, "i"', "\text{cofib}"] \arrow[dr, "f"] & \\ Z \arrow[r, "\simeq", "g'"'] & Y \end{tikzcd}$$

$f = g'i$

for g, g' htpy equivs,
p fibration
i cofibration.

In fact, we can be explicit with these constructions.

<u>Defn</u> The mapping path space P_f of f is the pullback

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y^I \arrow[r] \arrow[ur, phantom, "\lrcorner"] & Y \end{tikzcd}$$

$$\gamma \longmapsto \gamma(1)$$

Explicitly, $P_f = \{(x,\gamma) \in X \times Y^I \mid f(x) = \gamma(1)\}$.

<u>Prop</u>

$$(x,\gamma) \longmapsto x$$

$$P_f \rightleftarrows X \quad \text{are homotopy inverses}$$

$$(x, \mathrm{const}_{f(x)}) \longleftarrow x$$

and the composite $P_f \longrightarrow Y^I \longrightarrow Y$ is a fibration

$$(x,\gamma) \longmapsto \gamma \longmapsto \gamma(1)$$

<u>Pf Ideas</u> • $P_f \longrightarrow P_f$

$$(x,\gamma) \longmapsto (x, \mathrm{const}_{\gamma(1)})$$

homotopic to id_{P_f} by "reeling each path into its endpoint."

The composite $X \to X$ is id_X on the nose.

- For a lifting problem

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i"'] & P_f \arrow[d] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd} \qquad \begin{tikzcd} (x,\gamma) \arrow[d, mapsto] \\ \gamma(1) \end{tikzcd}$$

Define $\tilde{h}(z,t) =$ "$g(z)$ modified so that its path portion just goes $\gamma(t) \rightsquigarrow \gamma(1)$"

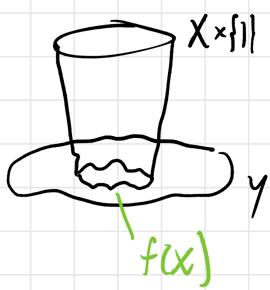
This works! $\square$

Defn The mapping cylinder of $f : X \to Y$ is the pushout

$$x \longmapsto (x,0)$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near end, "\ulcorner"] & X \times I \arrow[d] \\ Y \arrow[r] & M_f \end{tikzcd}.$$

Explicitly, $M_f = Y \amalg (X \times I) \big/ (x,0) \sim f(x)$.

$\circ \circ \circ \Big\{ \quad M_f =$ 

Prop $X \longrightarrow M_f$ is a cofibration and $Y \longrightarrow M_f$ is a htpy equiv.
$x \longmapsto [(x,1)]$ $\quad$ $y \longmapsto [y]$

Pf Moral exc. $\square$

Taken together, these props prove the theorem

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "\text{cofib}"'] \arrow[dr, "f"] & P_f \arrow[d, "\text{fib}"] \\ M_f \arrow[r, "\simeq"'] & Y \end{tikzcd}$$