Day 30

Learning Goals

- Category of pointed top spaces Top_*
- Smash - hom adjunction in Top_*
- Suspension and loops

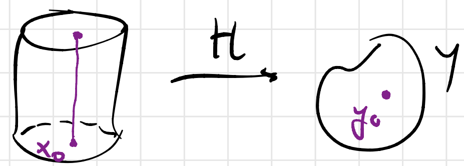
Defn The category of pointed spaces Top_* has objects pairs (X, x) for X a top space, $x \in X$ and morphisms $f : (X, x) \longrightarrow (Y, y)$ cts fns $f : X \to Y$ s.t. $f(x) = y$ — call such an f based.

A homotopy in Top_* is a map

$$H : I \times X \longrightarrow Y \qquad (\text{basepoints } x_0 \in X,\ y_0 \in Y)$$

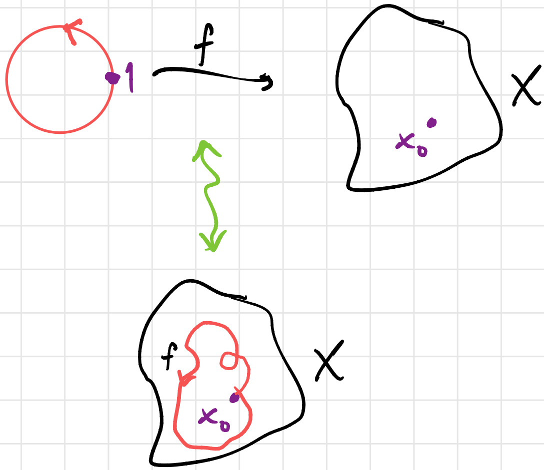
s.t. $h(t, x_0) = y_0 \quad \forall t \in I$.

$X = (X, x_0)$
$Y = (Y, y_0)$



The associated htpy category in $h\mathrm{Top}_*$ and we denote $h\mathrm{Top}_*(X, Y)$ by $[X, Y]_*$ or $[X, Y]$.

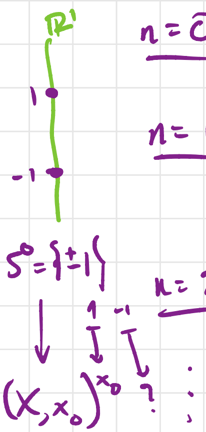
We have $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$



Give $S^n \subseteq \mathbb{R}^{n+1}$ the basepoint $e_1 = (1, 0, 0, \ldots, 0)$.

The functor $hTop_*(S^n, -) : Top_* \longrightarrow Set$
$(X, x_0) \longmapsto [S^n, X]$

is called the n-th homotopy set (actually group for $n \geq 1$, Abelian group for $n \geq 2$).



$\underline{n=0}$ $\pi_0(X, x) \cong$ path components of X
$[f] \longmapsto$ path cpt of X containing $f(-1)$

$\underline{n=1}$ $\pi_1(X, x) =$ fundamental group of X based at x

$\underline{n=2}$ $\pi_2(X, x) =$ second homotopy gp of X based at x
"cavities" in X?

$\pi_3(S^2) \cong \mathbb{Z}$

Have an adjunction

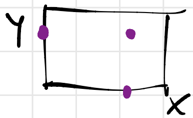
$$\begin{tikzcd} (\;)_+ : \mathrm{Top} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Top}_* : U \arrow[l, shift left] \end{tikzcd}$$

$$X \longleftarrow\!\shortmid (X, x)$$

$$Y \longmapsto Y_+ = (Y \amalg \{*\}, *)$$

Since U is a right adjoint, it preserves limits, so limits of diagrams of pointed spaces have the same underlying space as the limit of the corresponding diagram in Top.

E.g. $(X, x) \times (Y, y) = (X \times Y, (x, y))$,

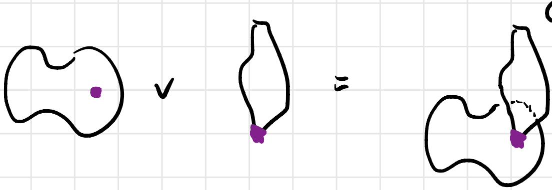


Colimits might be different!

Defn The wedge sum of ptd spaces is

$$(X, x) \vee (Y, y) := X \amalg Y /_{x \sim y}$$

categorical coproduct in Top_*



Defn The smash product of ptd spaces is

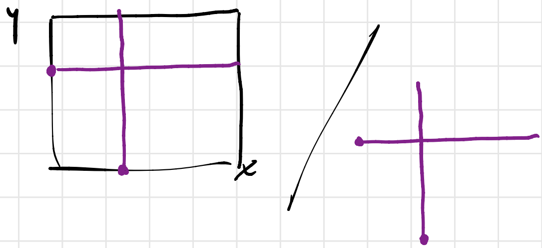
$$X \wedge Y := X \times Y / X \vee Y$$

not the categorical product in

Top_*

where $X \vee Y$ is identified with

$$(\{x_0\} \times Y) \cup (X \times \{y_0\}).$$



Thm For X locally compact Hausdorff, the functors $X \wedge -$, $(\ )^X$ form an adjunction

$$X \wedge - : \mathrm{Top}_* \rightleftarrows \mathrm{Top}_* : (\ )^X .$$

(Here $(Z, z_0)^{(X, x_0)}$ consists of based maps $X \to Z$ and is pointed by the constant function at z_0.)

Pf idea Descend from product-hom adjunction in Top.

$$f : Y \to Z^X \text{ based} \Rightarrow f(y_0) = \mathrm{const}_{z_0}.$$

$$\Rightarrow [f(y_0)](x) = z_0 \quad \forall x \in X$$

$\mathrm{Top}_*(X \wedge Y, Z) \cong \mathrm{Top}_*(Y, Z^X)$

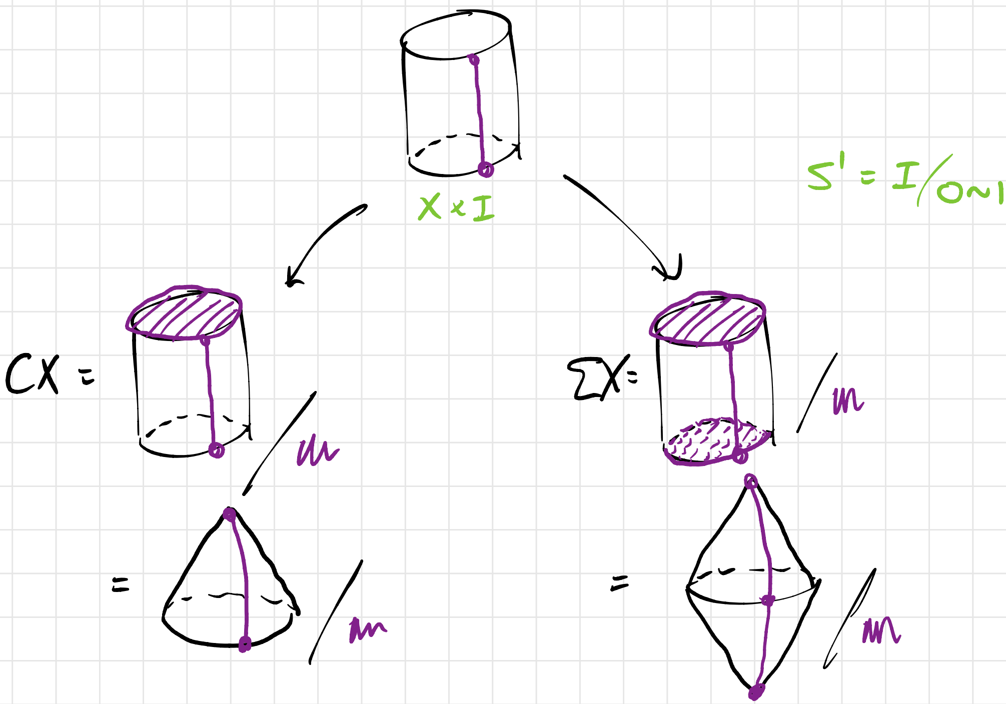
$$f(y) \text{ based } \forall y \in Y \Rightarrow [f(y)](x_0) = z_0 \quad \forall y \in Y.$$

Thus $\hat{f} : X \times Y \to Z$ is constant on $(\{x_0\} \times Y) \cup (X \times \{y_0\})$.

Defn The reduced cone and reduced suspension of (X, x_0) are

$$CX := X \wedge I \quad \text{and} \quad \Sigma X := X \wedge S^1$$

pted by 1



Let X^* denote the 1-point compactification of a space X, pted by $* = \infty$.

textbook for $X = Y = \mathbb{R}$

Thm $X^* \wedge Y^* \cong (X \times Y)^*$ Pf HW.

Cor $S^m \wedge S^n \cong S^{m+n}$ and $\Sigma S^n \cong S^{n+1}$.

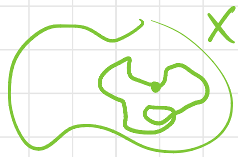
Pf Cor $S^n = (\mathbb{R}^n)^*$ and $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. $\square$

The based loops functor is $\Omega = (\ )^{S^1}$. Since $\Sigma \cong S^1 \wedge -$, we get the suspension-loop adjunction

$$\Sigma : \mathrm{Top}_* \underset{\perp}{\rightleftarrows} \mathrm{Top}_* : \Omega$$

$\Omega X = X^{S^1}$

$\Downarrow$



This descends to $h\mathrm{Top}_*$ so that

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

In particular,

$$\begin{aligned} \pi_{n+1} X &= [S^{n+1}, X] \\ &\cong [\Sigma S^n, X] \\ &\cong [S^n, \Omega X] \\ &= \pi_n \Omega X. \end{aligned}$$

$\pi_0 \Omega X \cong \pi_1 X$

<u>Cor</u> $\pi_n X$ is a group for $n \geq 1$.

<u>Pf</u> Induction + $\pi_{n+1} X \cong \pi_n \Omega X$. $\square$

<u>Note</u> $\pi_n X$ is Abelian for $n \geq 2$ (not necessarily for $n = 1$!)

One-pt compactification of X:

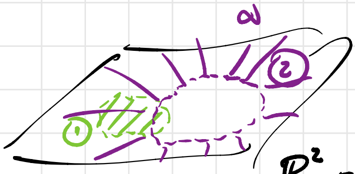
$X^* = X \amalg \{\infty\}$ as a set

opens in X^* are :

1. opens in X
2. complements of closed compact subsets of X

$$X^* \setminus K = (X \setminus K) \cup \{\infty\}$$

for $K \subseteq X$ compact closed.



$\mathbb{R}^2 = X$

$\shortparallel$

$\mathbb{C}$

$\mathbb{C}^* =$ Riemann sphere $\cong \mathbb{C}P^1$.