Day 29

Learning Goals

- Cylinder-free path adjunction and homotopies
- Def'n of fundamental groupoid Π_1 and group π_1.

Write $Y^X := \mathrm{Top}(X,Y)_{\text{exponential}}$ when X, Y are spaces such that $\mathrm{Top}(X,Y)$ admits a (unique!) exponential topology. Recall that this is the case for X loc cpt Hausdorff.

Let $I = [0,1]$ with std topology (subspace of $\mathbb{R}$).

There is an adjunction

$$I \times - : \mathrm{Top} \rightleftarrows \mathrm{Top} : (\ )^I$$

since I is compact H'ff.

Recall that a htpy b/w $f, g : X \to Y$ is

$$H : I \times X \longrightarrow Y \quad \text{s.t.} \quad H(0,-) = f, \ H(1,-) = g.$$

This corresponds to $\hat{H} : X \longrightarrow \underbrace{Y^I}_{\text{paths in } Y}$ where

$\hat{H}(x)$ is a path $f(x) \rightsquigarrow g(x)$.

Alternatively, the compact-open top is splitting, so we have an inj'n

$$\mathrm{Top}(X \times I, Y) \hookrightarrow \mathrm{Top}(I, \mathrm{Top}(X, Y)_{co})$$
$$H \longmapsto \text{a path from } f \text{ to } g$$

If X is loc cpt Hff this is a bij'n.

Recall that spaces X, Y are homotopy equivalent when $\exists f: X \to Y$, $g: Y \to X$ with $fg \simeq id_Y$, $gf \simeq id_X$.
Alternatively, $X \simeq Y$ iff $X \cong Y$ in hTop (where $\mathrm{hTop}(X,Y) := \mathrm{Top}(X,Y)/\simeq$).

Thm The map $\pi : X^I \to X$, $\gamma \mapsto \gamma(1)$ and $i : X \to X^I$, $x \mapsto \mathrm{const}_x$ are homotopy inverses.

Pf $i\pi : X^I \to X^I$, $\gamma \mapsto \mathrm{const}_{\gamma(1)}$. Define $H : X^I \times I \to X^I$

$$(\gamma, t) \longmapsto \left(\begin{array}{cc} I & s \\ \downarrow \gamma_t & \downarrow \\ X & \gamma(s+t-st) \end{array} \right)$$

Then $H(\gamma, 0) = \gamma$ and $H(\gamma, 1) = \mathrm{const}_{\gamma(1)}$ so $H : id_{X^I} \simeq i\pi$. Furthermore, $\pi i(x) = x$ so $\pi i = id_X$. $\square$

Thm The maps $i : X \to X \times I$, $x \mapsto (x, 1)$ and $p : X \times I \to X$, $(x,t) \mapsto x$ are htpy inverses.

Pf $pi = id_X$. $ip(x,t) = (x,1)$. **TB** Find a htpy $ip \simeq id_{X\times I}$.

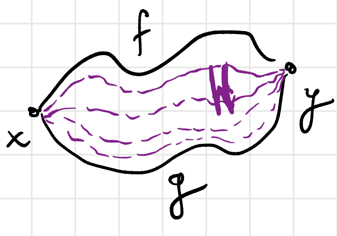
A $H : (X \times I) \times I \to X \times I$
$(x, t, s) \longmapsto (x, (1-s)t + s)$

Idea Think about paths in X up to homotopy.

Defn A groupoid is a category in which every morphism is an isomorphism.

Recall Groups correspond to groupoids w/a single object. Groupoids = "groups with many objects."

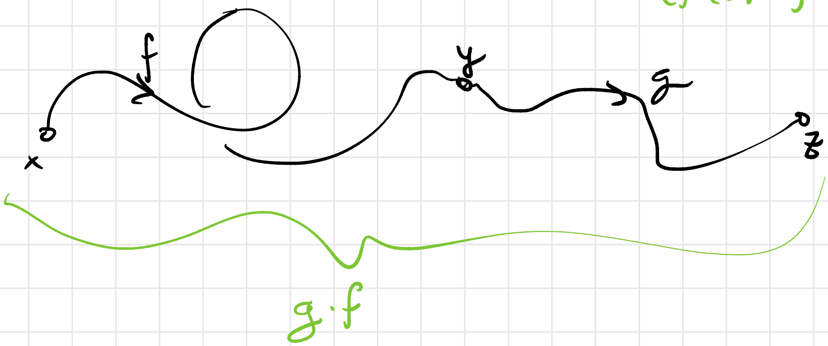
Defn Two paths $f, g : I \to X$ from x to y are path homotopic when $\exists$ htpy $H : I \times I \to X$ from f to g s.t. $h(0,t) = x$, $h(1,t) = y$ $\forall t$.



Defn The *fundamental groupoid* $\Pi_1 X$ of a space X is the category with objects pts of X, morphisms path htpy classes of paths $x \rightsquigarrow y$, composition

$$[f]\circ[g] = [\underbrace{f\cdot g}_{\text{concatenation}}]$$

$$f\cdot g\,(t) = \begin{cases} g(2t) & 0 \le t \le \tfrac{1}{2} \\ f(2t-1) & \tfrac{1}{2} \le t \le 1 \end{cases}$$



Note You should check that $[\mathrm{const}_x] = \mathrm{id}_x$, comp'n well-defined, associative; inverses by running paths in reverse.

We in fact get a functor

$$\begin{array}{rcl} \Pi_1 : \mathrm{Top} & \longrightarrow & \mathrm{Grpd} \\ X & \longmapsto & \Pi_1 X \quad [\gamma] \\ f\downarrow & \longmapsto & \downarrow \Pi_1 f \quad \downarrow \\ Y & \longmapsto & \Pi_1 Y \quad [f\gamma] \end{array}$$

Note In any cat C, for $x \in \mathrm{ob}\, C$, $\mathrm{Aut}_C(x) =$ $\{f \in C(x,x) \mid f \text{ iso}\}$ is a group. If C is a groupoid, $\mathrm{Aut}_C(x) = C(x,x)$.

Defn The *fundamental group* of X based at $x_0 \in X$ is $\pi_1(X, x_0) := \mathrm{Aut}_{\Pi_1 X}(x_0)$.

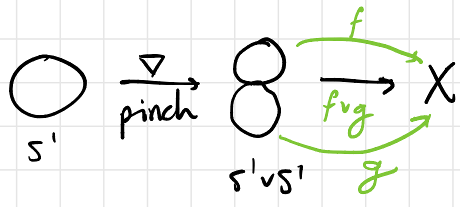
$=$ path htpy classes of paths $x_0 \rightsquigarrow x_0$

$=$ based htpy classes of maps $(S^1, (1,0)) \to (X, x_0)$.

How do we "multiply" two loops in X based at x_0?

- Concatenate: do one and then the other

or -



TPS Suppose x_0, x_1 are in the same path component of X. Prove that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.
What is the categorical version of this?