Day 28

Learning Goals

Exponential / compact-open topology

Motivation We have the product-hom adjunction

$$\mathrm{Set}(X\times Z, Y) \cong \mathrm{Set}(Z, \mathrm{Set}(X,Y)),$$

$g \longmapsto \hat{g}: Z \to \mathrm{Set}(X,Y)$

$z \longmapsto g(-,z): X \to Y$

Is there a functor $\mathrm{Top}(X,-): \mathrm{Top} \longrightarrow \mathrm{Top}$

s.t. $\mathrm{Top}(X\times Z, Y) \cong \mathrm{Top}(Z, \underbrace{\mathrm{Top}(X,Y)})$?

$g \longmapsto \hat{g}$

cts fns $X \to Y$ w/ a topology (function space)

Desiderata · $\mathrm{Top}(X,Y)$ coarse enough for $\hat{g}$ to be cts

· But if $\mathrm{Top}(X,Y)$ too coarse, then $\mathrm{Top}(Z, \mathrm{Top}(X,Y))$ will be too big!

Call a topology on $\mathrm{Top}(X,Y)$ splitting if g cts $\Rightarrow \hat{g}$ cts

conjoining if $\hat{g}$ cts $\Rightarrow g$ cts.

exponential if g cts $\Leftrightarrow \hat{g}$ cts.

Fact If there is an exponential topology on $\mathrm{Top}(X,Y)$ then it is unique. (Every splitting top is coarser than every conjoining topology.)

Based on "adjoint functor theorems", it's critical that

$$X \times - : \mathrm{Top} \longrightarrow \mathrm{Top} \quad \text{preserves colimits.}$$

True in Set (which has product-hom adjunction) but not for all $X \in \mathrm{ob}\,\mathrm{Top}$! (Top is not "Cartesian closed.")

Compact-Open Topology

Defn Let X, Y be spaces. For each compact $K \subseteq X$ and each open $U \subseteq Y$, define $S(K,U) := \{f \in \mathrm{Top}(X,Y) \mid fK \subseteq U\}$. Then form a subbasis for a topology on $\mathrm{Top}(X,Y)$ called the compact-open topology.

Note Subbasis for prod top on $\mathrm{Top}(X,Y)$ is formed by $S(F,U)$, F finite. I.e. prod top = "finite-open top" and compact open top is finer than this since finite $\Rightarrow$ compact.

X a space, Y a metric space

Recall A sequence (f_n) in $\mathrm{Top}(X,Y)_{\mathrm{prod}}$ converges to f iff it converges pointwise.

Thm X a compact space, Y a metric space

A sequence (f_n) in $\mathrm{Top}(X,Y)_{\text{cpt-open}}$ converges to f iff it converges uniformly

In fact, in this case, $\mathrm{Top}(X,Y)_{co}$ is metrized by

$$d(f,g) = \sup_{x\in X} d(f(x),g(x)).$$

(See Thm 5.4.)

Thm $\forall X,Y$, $\mathrm{Top}(X,Y)_{co}$ is splitting.

Pf Z a space, $g: X\times Z \to Y$ cts. For $\hat{g}: Z \to \mathrm{Top}(X,Y)$ cts, need to show $(\hat{g})^{-1}S(K,U) = \{z\in Z \mid g(K,z)\subseteq U\}$ is open in Z. Take $z\in (\hat{g})^{-1}S(K,U)$, so $z\in Z$, $g(K,z)\subseteq U$. By g cts, $g^{-1}U = \{(x,z) \mid g(x,z)\subseteq U\}\subseteq X\times Z$ open, and contains $K\times\{z\}$. By the Tube Lemma, $\exists$ open V,W with $K\subseteq V$, $z\in W$ with $K\times\{z\}\subseteq V\times W\subseteq g^{-1}U$

$\Rightarrow$ $z\in W\subseteq (\hat{g})^{-1}S(K,U)$ as needed. $\square$

Thm If X is locally cpt H'ff and Y is any space, then $\mathrm{Top}(X,Y)_{co}$ is exponential.

Recall Locally cpt H'ff $\Rightarrow$ $\forall$ open $U \subseteq X$ containing some x, $\exists\, V \subseteq X$ open with $\overline{V}$ cpt and

$$x \in V \subseteq \overline{V} \subseteq U.$$

Lemma A topology on $Top(X, Y)$ is conjoining iff $eval : X \times Top(X,Y) \longrightarrow Y$ is cts.

Pf 5.1 in book. $\square$

Pf Thm Only need to check $Top(X,Y)_{co}$ is conjoining i.e. $eval: X \times Top(X,Y)_{co} \longrightarrow Y$ is cts. Let $(x,f) \in X \times Top(X,Y)$ and let $U \subseteq Y$ open containing $eval(x,f) = f(x)$. Since f cts, $f^{-1}U \subseteq X$ open containing x. Since X loc cpt H'ff, $\exists\, V \subseteq X$ with $K := \overline{V}$ cpt and $x \in V \subseteq K \subseteq f^{-1}U$. This implies

$$f(x) \in fK \subseteq U.$$

Then $V \times S(K,U) \subseteq Top(X,Y)_{co}$ is open with $(x,f) \in V \times S(K,U)$ and $eval(V \times S(K,U)) \subseteq U$. $\square$