

Day 27

Learning Goals

- Adjoints to the forgetful functor $U: \text{Top} \rightarrow \text{Set}$
- Stone-Čech compactification

Let $U: \text{Top} \rightarrow \text{Set}$ be the forgetful functor.

$$\begin{aligned}(X, \tau) &\mapsto X \\ f &\mapsto f\end{aligned}$$

Consider $\text{Set}(X, UY) = (UY)^X = \text{all fns } X \rightarrow Y$.

Q Is there a top space (built from X) such that all fns from this space to Y are in bij'n with $\text{Set}(X, UY)$?

A of course! $(X, \tau_{\text{disc}}) =: DX$. We have a functor

$$\begin{aligned}D: \text{Set} &\rightarrow \text{Top} \\ X &\mapsto (X, \tau_{\text{disc}}) \\ f &\mapsto f\end{aligned}$$

and $D: \text{Set} \rightleftarrows \text{Top}: U$
is an adjoint pair via

$$\begin{aligned}\text{Top}(DX, Y) &\cong \text{Set}(X, UY) \\ f &\leftrightarrow f\end{aligned}$$

Recall the indiscrete (concrete, chaotic, ...) topology on X , $\tau_{\text{ind}} = \{\emptyset, X\}$, and write $I: \text{Set} \rightarrow \text{Top}$

$$\begin{array}{ccc} \text{for the functor } X & \longmapsto & IX = (X, \tau_{\text{ind}}) \\ f \downarrow & \longmapsto & \downarrow If = f \\ Y & \longmapsto & IY \end{array}$$

Which functions $f: X \rightarrow IY$ are cts? Just need $f^{-1}\emptyset = \emptyset$ and $f^{-1}y = X$ open in X , so all functions into IY are cts. I.e.

$$\text{Set}(UX, Y) \cong \text{Top}(X, IY)$$

$$f \longleftrightarrow f$$

so $U: \text{Top} \rightleftarrows \text{Set}: I$ is an adjoint pair:
 U admits both left and right adjoints!

$$\begin{array}{ccc} & \text{Top} & \\ & \downarrow & \\ D & \left(\begin{array}{c} \dashv \quad U \quad \dashv \\ \downarrow \\ \end{array} \right) & I \\ & \uparrow & \\ & \text{Set} & \end{array}$$

This has nice implications for (co)continuity of U :

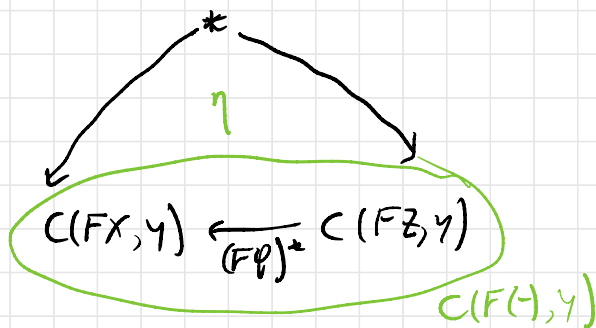
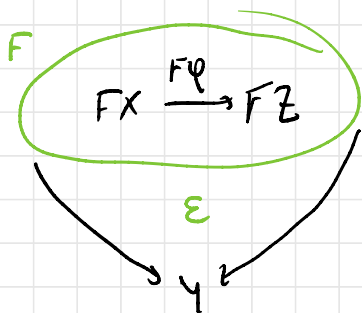
Thm If $L: C \rightarrow D$ has a right adjoint, then L is cocontinuous. If $R: D \rightarrow C$ has a left adjoint, then R is continuous.

Cor $U: \text{Top} \rightarrow \text{Set}$ is cts and cocts.

Lemma For any diagram $F: D \rightarrow C$ with $\text{colim } F$ in C ,
 $C(\text{colim } F, Y) \cong \lim C(F(-), Y)$ naturally in Y .

Pf Idea $C(\text{colim } F, Y) \cong \text{Nat}_{D \rightarrow C}(F, Y)$

$$\lim C(F(-), Y) \cong \text{Nat}_{D^{\text{op}} \rightarrow \text{Set}}(*, C(F(-), Y))$$



$$\epsilon \longmapsto \left\{ \alpha \mapsto \epsilon_X \right\}_{X \in \text{ob } D}$$

$$\left\{ \eta_X \right\}_{X \in \text{ob } D} \longleftarrow \eta$$



Pf Thm Suppose L admits a right adjoint. Then

$$\begin{aligned} D(L(\text{colim } F), Y) &\cong C(\text{colim } F, RY) \\ &\cong \lim C(F(-), RY) \\ &\cong \lim C(LF(-), Y) \\ &\cong D(\text{colim } LF, Y) \end{aligned}$$

$\Rightarrow L(\operatorname{colim} F)$ satisfies the univ prop of colim LF.

By uniqueness of colims, $L(\operatorname{colim} F) \cong \operatorname{colim} F$.

Other case similar. □

Takeaway It was no coincidence that the set underlying product, quotient, pullback, etc of spaces matched the corresponding (co)limit of sets:

U preserves (co)limits.

Stone-Čech compactification

going to give the big ideas but not all details!

Let \mathbf{CH} denote the category of compact Hausdorff spaces and cts fns (i.e. \mathbf{CH} the full subcat of \mathbf{Top} w/ objects c't H'ff spaces).

We have a forgetful functor $U: \mathbf{CH} \hookrightarrow \mathbf{Top}$

which is the embedding of

$$\begin{array}{ccc} X & \hookrightarrow & X \\ f & \mapsto & f \end{array}$$

\mathbf{CH} in \mathbf{Top} . (fully faithful)

Thm U admits a left adjoint $\beta: \mathbf{Top} \rightarrow \mathbf{CH}$, the Stone-Čech compactification.

Let's explore the implications:

$$\text{CH}(\beta X, Y) \cong \text{Top}(X, \text{UY}) = \text{Top}(X, Y).$$

for X any space and Y compact H'ff.

So if U admits a left adjoint β is determined since we know the maps out of βX (Yoneda!).

Two paths forward:

① Use an adjoint functor theorem to prove such an adjoint exist

② Construct β . ← we'll do this!

see 5.4

βX is the space of ultrafilters on X

For a set X , let $\beta X = \{ \mathcal{F} \mid \mathcal{F} \text{ an ultrafilter on } X \}$
(forgetting the previous meaning of β).

Give βX a topology with basic open sets of the form $\hat{A} := \{ \mathcal{F} \in \beta X \mid A \in \mathcal{F} \}$ for $A \subseteq X$.

We actually have $\widehat{A \cap B} = \hat{A} \cap \hat{B}$ so this is a basis.

For $f: X \rightarrow Y$ a function, we have $\beta f = f_*: \beta X \rightarrow \beta Y$.

$$\text{Moreover, } (\beta f)^{-1}(\hat{B}) = \widehat{f^{-1}B}$$

so βf is cts.

↑ pushforward along f

There is a function $p_X: X \rightarrow \beta X$
 $x \mapsto p_X(x) = \{A \in X \mid x \in A\}$
 the principal uf for x .

Prop p_X is injective and $p_X(X) \subseteq \beta X$ is dense.

Pf Injectivity is clear since the singleton sets containing x, y ($x \neq y$) distinguish $p_X(x), p_X(y)$.

For density, must show that every open in βX contains a principal uf. Suffices to do so for basic opens \hat{A} ,

$A \in X$. If $\mathcal{F} \in \hat{A}$, then $A \neq \emptyset$ and hence $\exists x \in A$.

Thus $A \in p_X(x) \Rightarrow p_X(x) \in \hat{A}$. □

Thm For any set X , βX is compact Hausdorff.

Pf H'ff: Take distinct $\mathcal{F}, \mathcal{G} \in \beta X$. Then

$\exists A \in X, A \in \mathcal{F}, A \notin \mathcal{G}$. Since \mathcal{G} is max'l, $X \setminus A \in \mathcal{G}$.

Then $\hat{A}, \widehat{X \setminus A}$ are disjoint opens containing \mathcal{F}, \mathcal{G} resp.

Compact: (challenging?) moral exercise. □

Recall X H'ff \Leftrightarrow ufs have at most one limit
 X cts \Leftrightarrow ufs have at least one limit

X cpt H'ff \Leftrightarrow cfs have exactly one limit.

So for X cpt H'ff, get a function

$$\begin{aligned} \varepsilon_x: \beta U X &\longrightarrow X \\ \mathcal{F} &\longmapsto \lim \mathcal{F} \end{aligned}$$

Prop For X cpt H'ff, ε_x is cts.

By taking $\eta_x = \eta_x: X \rightarrow U\beta X$, it turns out that η, ε are the unit and counit for the adjunction $U: CH \rightleftharpoons Top: \beta$!