Day 27

Learning Goals

- Adjoints to the forgetful functor $U: \mathrm{Top} \to \mathrm{Set}$
- Stone-Čech compactification

Let $U: \mathrm{Top} \to \mathrm{Set}$ be the forgetful functor.

$$(X, \mathcal{T}) \mapsto X$$
$$f \mapsto f$$

Consider $\mathrm{Set}(X, UY) = (UY)^X$ = all fns $X \to Y$.

Q Is there a top space (built from X) such that cts fns from this space to Y are are in bij'n with $\mathrm{Set}(X, UY)$?

A of course! $(X, \mathcal{T}_{disc}) =: DX$. We have a functor

$$D: \mathrm{Set} \to \mathrm{Top}$$
$$X \mapsto (X, \mathcal{T}_{disc})$$
$$f \mapsto f$$

and $D: \mathrm{Set} \rightleftarrows \mathrm{Top} : U$ is an adjoint pair via

$$\mathrm{Top}(DX, Y) \cong \mathrm{Set}(X, UY).$$
$$f \longleftrightarrow f$$

Recall the indiscrete (concrete, chaotic, ...) topology on X, $\mathcal{T}_{ind} = \{\emptyset, X\}$, and write $I: \mathrm{Set} \to \mathrm{Top}$

for the functor

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & IX = (X, \mathcal{T}_{ind}) \arrow[d, "If = f"] \\ Y \arrow[r, mapsto] & IY \end{tikzcd}$$

Which functions $f: X \to IY$ are cts? Just need $f^{-1}\varnothing = \varnothing$ and $f^{-1}Y = X$ open in X, so <u>all</u> functions into IY are cts. I.e.

$$\mathbf{Set}(UX, Y) \cong \mathbf{Top}(X, IY)$$
$$f \longleftrightarrow f$$

so $U: \mathbf{Top} \rightleftarrows \mathbf{Set} : I$ is an adjoint pair:

U admits both left and right adjoints!

$$\begin{tikzcd} \mathbf{Top} \arrow[d, "U" description] \\ \mathbf{Set} \arrow[u, bend left=60, "D"] \arrow[u, bend right=60, "I"'] \end{tikzcd} \qquad D \dashv U \dashv I$$

This has nice implications for (co)continuity of U:

Thm If $L: \mathcal{C} \to \mathcal{D}$ has a right adjoint, then L is cocontinuous. If $R: \mathcal{D} \to \mathcal{C}$ has a left adjoint, then R is continuous.

Cor $U: \mathbf{Top} \to \mathbf{Set}$ is cts and cocts.

Lemma For any diagram $F: D \to C$ with $\mathrm{colim}\, F$ in C,

$$C(\mathrm{colim}\, F, Y) \cong \lim C(F(-), Y) \text{ naturally in } Y.$$

Pf Idea $$C(\mathrm{colim}\, F, Y) \cong \mathrm{Nat}_{D \to C}(F, Y)$$

$$\lim C(F(-), Y) \cong \mathrm{Nat}_{D^{op} \to Set}(*, C(F(-), Y))$$

F

$$\begin{tikzcd} FX \arrow[r, "F\varphi"] \arrow[dr] & FZ \arrow[d] \\ & Y \end{tikzcd}$$

ε

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ C(FX, Y) & & C(FZ, Y) \arrow[ll, "(F\varphi)^*"] \end{tikzcd}$$

η

$C(F(-), Y)$

$$\varepsilon \longmapsto \{* \mapsto \varepsilon_X\}_{X \in \mathrm{ob}\, D}$$

$$\{\eta_X *\}_{X \in \mathrm{ob}\, D} \longmapsfrom \eta$$

$\square$

Pf Thm Suppose L admits a right adjoint. Then

$$\begin{aligned} D(L(\mathrm{colim}\, F), Y) &\cong C(\mathrm{colim}\, F, RY) \\ &\cong \lim C(F(-), RY) \\ &\cong \lim C(LF(-), Y) \\ &\cong D(\mathrm{colim}\, LF, Y) \end{aligned}$$

$\Rightarrow L(\operatorname{colim} F)$ satisfies the univ prop of $\operatorname{colim} LF$.

By uniqueness of colims, $L(\operatorname{colim} F) \cong \operatorname{colim} F$.

Other case similar. $\square$

Takeaway It was no coincidence that the set underlying product, quotient, pullback, etc of spaces matched the corresponding (co)limit of sets:

U preserves (co)limits.

Stone-Čech compactification going to give the big ideas but not all details!

Let CH denote the category of compact Hausdorff spaces and cts fns (i.e. CH the full subcat of Top w/ objects cpt H'ff spaces).

We have a forgetful functor $U : \mathrm{CH} \hookrightarrow \mathrm{Top}$ which is the embedding of CH in Top. (fully faithful)

$$X \longmapsto X$$
$$f \longmapsto f$$

Thm U admits a left adjoint $\beta : \mathrm{Top} \to \mathrm{CH}$, the **Stone-Čech compactification**.

Let's explore the implications:

$$\mathrm{CH}(\beta X, Y) \cong \mathrm{Top}(X, UY) = \mathrm{Top}(X, Y).$$

for X any space and Y compact H'ff.

So if U admits a left adjoint β is determined since we know the maps out of βX (Yoneda!).

Two paths forward:

① Use an adjoint functor theorem to prove such an adjoint exist — see 5.4

② Construct β. ⟵ we'll do this!

βX is the space of ultrafilters on X

For a set X, let $\beta X = \{ \mathcal{F} \mid \mathcal{F} \text{ an ultrafilter on } X \}$ (forgetting the previous meaning of β).

Give βX a topology with basic open sets of the form $\hat{A} := \{ \mathcal{F} \in \beta X \mid A \in \mathcal{F} \}$ for $A \subseteq X$.

We actually have $\widehat{A \cap B} = \hat{A} \cap \hat{B}$ so this is a basis.

For $f : X \to Y$ a function, we have $\beta f := f_* : \beta X \to \beta Y$.

pushforward along f

Moreover, $(\beta f)^{-1}(\hat{B}) = \widehat{f^{-1} B}$

so βf is cts.

There is a function $p_X : X \to \beta X$

$$x \longmapsto p_X(x) = \{A \subseteq X \mid x \in A\}$$

the principal uf for x.

Prop p_X is injective and $p_X(X) \subseteq \beta X$ is dense.

Pf Injectivity is clear since the singleton sets containing x, y $(x \neq y)$ distinguish $p_X(x)$, $p_X(y)$.

For density, must show that every open in βX contains a principal uf. Suffices to do so for basic opens $\hat{A}$, $A \subseteq X$. If $\mathcal{F} \in \hat{A}$, then $A \neq \emptyset$ and hence $\exists x \in A$. Thus $A \in p_X(x) \implies p_X(x) \in \hat{A}$. $\square$

Thm For any set X, βX is compact Hausdorff.

Pf H'ff: Take distinct $\mathcal{F}, \mathcal{G} \in \beta X$. Then $\exists A \subseteq X$, $A \in \mathcal{F}$, $A \notin \mathcal{G}$. Since $\mathcal{G}$ is max'l, $X \setminus A \in \mathcal{G}$. Then $\hat{A}$, $\widehat{X \setminus A}$ are disjoint opens containing $\mathcal{F}, \mathcal{G}$ resp.

Compact: (challenging?) moral exercise. $\square$

Recall X H'ff $\iff$ ufs have at most one limit

X cts $\iff$ ufs have at least one limit

X cpt H'ff $\iff$ ufs have exactly one limit.

So for X cpt H'ff, get a function

$$\begin{aligned} \varepsilon_X : \beta U X &\longrightarrow X \\ \mathcal{F} &\longmapsto \lim \mathcal{F} \end{aligned}$$

Prop For X cpt H'ff, ε_X is cts.

By taking $\eta_X = p_X : X \longrightarrow U\beta X$, it turns out that η, ε are the unit and counit for the adjunction $U : CH \rightleftarrows Top : \beta$!