Day 26

Learning Goals

- Bases of vector spaces and the free-forgetful adjunction
- Adjunctions in general
- Unit & counit of an adjunction
- Product-hom adjunction.

Recall from linear algebra that linear transformations are determined by their action on a basis.

More precisely, take V, W vector spaces, B a basis of V. Given lin trans $f: V \to W$, get a function $f|_B : B \to W$ by restriction. And given $g: B \to W$ a function, we can extend g linearly to a lin trans

$$\tilde{g} : V \longrightarrow W$$
$$\sum \lambda_i b_i \longmapsto \sum \lambda_i g(b_i).$$

Diagrammatically,

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d] & W \\ V \arrow[ur, dashed, "\exists! \tilde{g}"'] & \end{tikzcd}$$

This is an odd diagram since B is a set, g is a function, V, W vs, $\tilde{g}$ lin trans.

But every vs has an underlying set and every lin trans is a function b/w underlying sets satisfying particular properties.

Let $U : \mathrm{Vect}_k \to \mathrm{Set}$ be the forgetful (underlying) functor. The above diagram is really

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d] & UW \\ UV \arrow[ur, dashed, "\exists! U\tilde{g}"'] & \end{tikzcd}$$

We thus get a bijection

$$\mathrm{Set}(B, UW) \cong \mathrm{Vect}_k(V, W).$$

We can functorially assign vector spaces to sets via the free functor

$$F : \mathrm{Set} \longrightarrow \mathrm{Vect}_k$$

$$B \longmapsto FB = k^{\oplus B} = \{B\text{-tuples in } k \text{ with almost all values } 0\}$$

$$= kB, \text{ formal linear combos of } b \in B$$

$$\begin{array}{ccc} B & & FB \quad b \\ f\downarrow & \longmapsto & \downarrow \quad \ \ \downarrow \\ C & & FC \quad 1\cdot f(b) \end{array} \quad \text{extended linearly.}$$

Since B is a basis of FB, $\circledast$ becomes

$$\mathrm{Vect}_k(FB, W) \cong \mathrm{Set}(B, UW).$$
$$f \longmapsto f|_B$$

This iso is natural in B and W:

$$\mathrm{Vect}_k(F(-), W) \cong \mathrm{Set}(-, UW)$$

and $$\mathrm{Vect}_k(FB, -) \cong \mathrm{Set}(B, U(-)).$$

This makes $F : \mathrm{Set} \rightleftarrows \mathrm{Vect}_k : U$ an adjoint pair of functors:

Defn An *adjunction* between categories $\mathcal{C}, \mathcal{D}$ is a pair of functors $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$ together with a binatural isomorphism

$$\hat{(\ )} : \mathcal{D}(LX, Y) \xrightarrow{\cong} \mathcal{C}(X, RY)$$
$$f \longmapsto \hat{f} = \text{adjunct of } f.$$

We say L is left adjoint to R, R is right adjoint to L, (L, R) is an adjoint pair.

Write $L \dashv R$ or $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ or
$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "L"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{D} \arrow[u, shift right, "R"'] \end{tikzcd}$$

Binatural: Natural isos

$$\varphi_{\bullet, Y} : \mathcal{D}(L(\ ), Y) \xrightarrow{\cong} \mathcal{C}(-, RY)$$
$$\varphi_{X, \bullet} : \mathcal{D}(LX, -) \xrightarrow{\cong} \mathcal{C}(X, R(-))$$

for all $X \in \text{ob}\, \mathcal{C}$, $Y \in \text{ob}\, \mathcal{D}$.

E.g. For $F : \mathbf{Set} \rightleftarrows \mathbf{Vect}_k : U$ the natural iso is

$$f \longmapsto f|_B$$
$$\tilde{g} \longmapsfrom g.$$

When $Y = LX$, get

$$\mathcal{D}(LX, LX) \xrightarrow{\cong} \mathcal{C}(X, RLX)$$
$$\text{id}_{LX} \longmapsto \widehat{\text{id}_{LX}} =: \eta_X$$

These assemble into a natural trans

$$\eta : \text{id}_{\mathcal{C}} \Longrightarrow RL$$
$$\eta_X : X \longrightarrow RLX \quad \forall X \in \text{ob}\, \mathcal{C}$$

called the unit of the adjunction (L, R).

Similarly, with $X = RY$ get

$$D(LRY, Y) \xrightarrow{\cong} C(Y, Y)$$
$$\varepsilon_Y := \widehat{id_Y} \longmapsto id_Y$$

assembling into a natural trans

$$\varepsilon : LR \Rightarrow id_D$$
$$\varepsilon_Y : LRY \longrightarrow Y \quad \forall Y \in ob\, D$$

called the counit of the adjunction (L, R).

E.g. For $F : Set \rightleftarrows Vect_k : U$,

$$Vect_k(FB, FB) \longrightarrow Set(B, UFB)$$
$$id_{FB} \longmapsto id_{FB}|_B = \eta_B$$

$$Vect_k(FUV, V) \longrightarrow Set(UV, UV)$$
$$\varepsilon_V = \begin{pmatrix} \sum \lambda_i v_i \\ \text{formal} \\ \downarrow \\ \sum \lambda_i v_i \\ \text{in } V \end{pmatrix} \longmapsto id_{UV}$$

(See text for how to recover $\hat{(\;)}$ from η, ε.)

Product-Hom adjunction in Set

For $X \in \mathrm{ob}\,\mathbf{Set}$, consider

$$L = X \times - : \mathbf{Set} \to \mathbf{Set}$$

$$R = \mathbf{Set}(X, -) : \mathbf{Set} \to \mathbf{Set}.$$

Have

$$\begin{array}{ccc} \mathbf{Set}(LZ, Y) = Y^{X \times Z} & & f : X \times Z \to Y \\ \cong \big\downarrow \hat{(\ )} & & \big\downarrow \\ \mathbf{Set}(Z, RY) = (Y^X)^Z & & \hat{f} : Z \to Y^X \end{array}$$

$$z \longmapsto \left(\begin{array}{cc} X & x \\ \downarrow & \downarrow \\ Y & f(x,z) \end{array} \right)$$

The inverse is given by

$$\begin{array}{l} \hat{g} : X \times Z \to Y \\ \qquad (x, z) \longmapsto (g(z))(x) \\ \big\uparrow \\ g : Z \longrightarrow Y^X . \end{array}$$

Unit

$$\eta_X : X \longrightarrow RLX = (X \times X)^X .$$

$$z \longmapsto \left(\begin{array}{cc} X & x \\ \downarrow & \downarrow \\ X \times X & (x, z) \end{array} \right)$$

Counit

$$\varepsilon_Y : LRY = X \times Y^X \xrightarrow{\text{eval}} Y .$$

$$(x, f) \longmapsto f(x)$$

E.g. In Vect_k, it is <u>not</u> the case that

$$\mathrm{Vect}_k(U \times V, W) \cong \mathrm{Vect}_k(V, \mathrm{Vect}_k(U, W)).$$

Must replace $\times$ with a new operation: tensor product $U \otimes_k V$.

This leads to the "tensor-hom adjunction" in linear algebra.

Need either to construct $U \otimes_k V$ so it fits in the adjunction or have some reason why $\mathrm{Vect}_k(-, W)$ "admits a left adjoint."