Day 25

Learning Goals

- More (co)limit examples
- (Co)completeness

Defn Given parallel morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in a cat C,

$\lim\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) =:$ equalizer of f, g, and

$\operatorname{colim}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) =:$ coequalizer of f, g.

I.e. $E = \lim\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) \xrightarrow{\eta} X \underset{g}{\overset{f}{\rightrightarrows}} Y$

then $\boxed{f\eta = g\eta}$ and

$$\begin{tikzcd}
S \arrow[d, dashed, "\exists!"'] \arrow[dr] & & \\
E \arrow[r, "\eta"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}.$$

For the coequalizer C:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\varepsilon"'] \arrow[dr] & C \arrow[d, dashed, "\exists!"] \\
& & S
\end{tikzcd}$$

E.g. $\cdot$ In $C = \mathrm{Gp}$, $\lim\left(G \overset{e}{\underset{f}{\rightrightarrows}} H\right) = \ker(f)$

(where e = const map $g \mapsto e$, id elt of H).

$\cdot$ In $C = \mathrm{Vect}_k$, $\lim\left(V \overset{0}{\underset{f}{\rightrightarrows}} W\right) = \ker(f)$.

$\cdot$ In $C = \mathrm{Set}$ or Top,

$$\lim\left(X \overset{f}{\underset{g}{\rightrightarrows}} Y\right) = \{x \in X \mid f(x) = g(x)\}$$

$\cdot$ In $C = \mathrm{Gp}$, $\mathrm{colim}\left(G \overset{e}{\underset{f}{\rightrightarrows}} H\right) = \mathrm{coker}(f)$

$$= H/\mathrm{im}(f)$$

$\cdot$ In $C = \mathrm{Vect}_k$, $\mathrm{colim}\left(V \overset{0}{\underset{f}{\rightrightarrows}} W\right) = \mathrm{coker}(f)$

$$= W/\mathrm{im}(f)$$

$\cdot$ In $C = \mathrm{Set}$ or Top,

$$\mathrm{colim}\left(X \overset{f}{\underset{g}{\rightrightarrows}} Y\right) = Y/_{f(x) \sim g(x)}.$$

(Co)completeness

Defn A cat C is complete when it contains all limits of small diagrams, cocomplete when it contains all colimits of small diagrams.

E.g. Set and Top are bicomplete.

- In Set, $\lim(F: D \to \mathrm{Set})$

$$= \left\{ (x_d) \in \prod_{d \in \mathrm{Ob}\, D} Fd \;\middle|\; \begin{array}{l} \text{proj'n maps} \\ \text{form a cone over } F \end{array} \right\}.$$

In Top, $\lim(F: D \to \mathrm{Top})$ is the limit in Set w/ subspace top inside product $\prod Fd$.

- In Set, $\mathrm{colim}(F: D \to \mathrm{Top})$

$$= \coprod_{d \in \mathrm{Ob}\, D} Fd \Big/ \sim$$

for $\sim$ the coarsest equiv reln making this a cone under F.

In Top, give this the coprod then quotient top.

Thm If C has small products and equalizers, then C is complete. If C has small coproducts and coequalizers, then C is cocomplete.

Pf Idea for Limits Given $F: D \to C$,

$$\lim F = \lim \left(\prod_{d \in \mathrm{Ob}\, D} Fd \overset{s}{\underset{t}{\rightrightarrows}} \prod_{\substack{d \xrightarrow{f} e \\ \text{in } \mathrm{Mor}\, D}} Fe \right)$$

where $s\big((x_d)_{d \in \mathrm{Ob}\, D}\big) = \big(\pi_{\mathrm{cod}(f)} (x_d)_{d \in \mathrm{Ob}\, D}\big)_{f \in \mathrm{Mor}\, D}$

$$t\big((x_d)_{d \in \mathrm{Ob}\, D}\big) = \big(Ff(x_{\mathrm{dom}(f)})\big)_{f \in \mathrm{Mor}\, D}.$$

The universal cone is given by composing equalizer $\longrightarrow \prod_{d \in \mathrm{Ob}\, D} Fd$ with proj'n maps. $\square$