Day 24

Learning goals

- Examples of (co)limits
- Uniqueness of (co)limits

Terminal and initial objects

Suppose $D = \emptyset$, the cat with no objects, no morphisms.

A functor $\emptyset \to C$ is an *empty diagram*.

When they exist,

$\lim(\emptyset \to C) =: * =$ *terminal object*

$\operatorname{colim}(\emptyset \to C) =: \emptyset =$ *initial object*

The terminal objects accepts a unique morphism from every object in C, and the initial object maps uniquely to each object in C.

Category	Initial	Terminal
Set	$\emptyset$	singleton set
Top	$\emptyset$	singleton space
Gp	$\{e\}$	$\{e\}$
FinVect_k	$\{0\}$	$\{0\}$

We have already seen (co)products as (co)limits of diagrams of shape $\bullet \quad \bullet$. We also saw pullbacks as limits of shape

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Pushouts

A functor from $\bullet \leftarrow \bullet \rightarrow \bullet$ is a diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

with colimit the pushout of X and Y along f, g, denoted $X \amalg_Z Y$ or $X {}_f\!\amalg_g Y$.

In Set, $X \amalg_Z Y = X \amalg Y / f(z) \sim g(z)$

equiv rel'n gen'd by $f(z) \sim g(z)$ for $z \in Z$.

Same in Top (w/ quotient topology)

We denote a pushout diagram as

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ Y \arrow[r] & X \amalg_Z Y \end{tikzcd}$$

.

As a special case, in Set and Top,

$$\begin{tikzcd} A\cap B \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, hook] \\ B \arrow[r, hook] & A\cup B \end{tikzcd}$$

Indeed,

$$\begin{tikzcd} A\cap B \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \arrow[ddr, bend left, "f"] & \\ B \arrow[r, hook] \arrow[drr, bend right, "g"'] & A\cup B \arrow[dr, dashed, "\exists! h"] & \\ & & S \end{tikzcd}$$

$$h(x) = \begin{cases} f(x) & x\in A \\ g(x) & x\in B \end{cases}$$

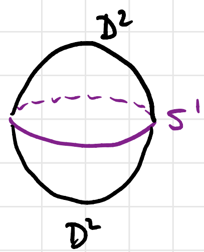
Well-defined since f, g agree on $A\cap B$!

Pushouts also permit the construction of spaces by attaching cells. Attach D^n to X along its boundary S^{n-1} via

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ D^n \arrow[r] & X \amalg_f D^n \end{tikzcd}.$$

E.g.

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & D^2 \arrow[d] \\ D^2 \arrow[r] & S^2 \end{tikzcd}$$



Inverse and direct limits

The limit of a diagram of shape $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$ is called an *inverse limit*.

In Set,

$$\lim\left(X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \cdots\right)$$

is the subset of $\prod_{i\geq 1} X_i$ consisting of $(x_1, x_2, \ldots) \in \prod X_i$ s.t. $f_i(x_{i+1}) = x_i$ together with proj'n maps.

In Top, same set, endowed with subspace top inside $\prod X_i$.

The colimit of a diagram of shape $\bullet \to \bullet \to \bullet \to \cdots$ is called a *directed colimit*.

E.g. In AbGp,

$$\begin{tikzcd} \operatorname{colim}\big(\mathbb{Z} \arrow[r, "n"] \arrow[dr, "\cdot 1"'] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "\cdot \frac{1}{n}"] & \mathbb{Z} \arrow[r] \arrow[dl, "\cdot \frac{1}{n^2}"] & \cdots \big) \\ & = \mathbb{Z}[1/n] & & \end{tikzcd}$$

Limits are unique up to unique iso:

Suppose L and L' are limits of a diagram $F: D \to C$.

Then

$$\begin{tikzcd}
 & L' \arrow[dl, bend right, "\eta'_\bullet"'] \arrow[d, dashed, bend left, "\exists! h'"] \\
F_\bullet & L \arrow[l, "\eta_\bullet"] \arrow[u, dashed, bend left, "\exists! h"]
\end{tikzcd}$$

since both $\eta_\bullet$ and $\eta'_\bullet$ are terminal cones over F.

But now $h'h$ makes the diagram

$$\begin{tikzcd}
 & L \arrow[dl, bend right, "\eta_\bullet"'] \arrow[d, "h'h"] \\
F_\bullet & L \arrow[l, "\eta_\bullet"]
\end{tikzcd}$$

commute.

So does id_L, so by uniqueness, $h'h = id_L$. Similarly, $hh' = id_{L'}$. Thus h, h' are inverse isomorphisms, necessarily the only ones compatible with $\eta_\bullet, \eta'_\bullet$. $\square$

Similarly, colimits are unique up to unique iso.