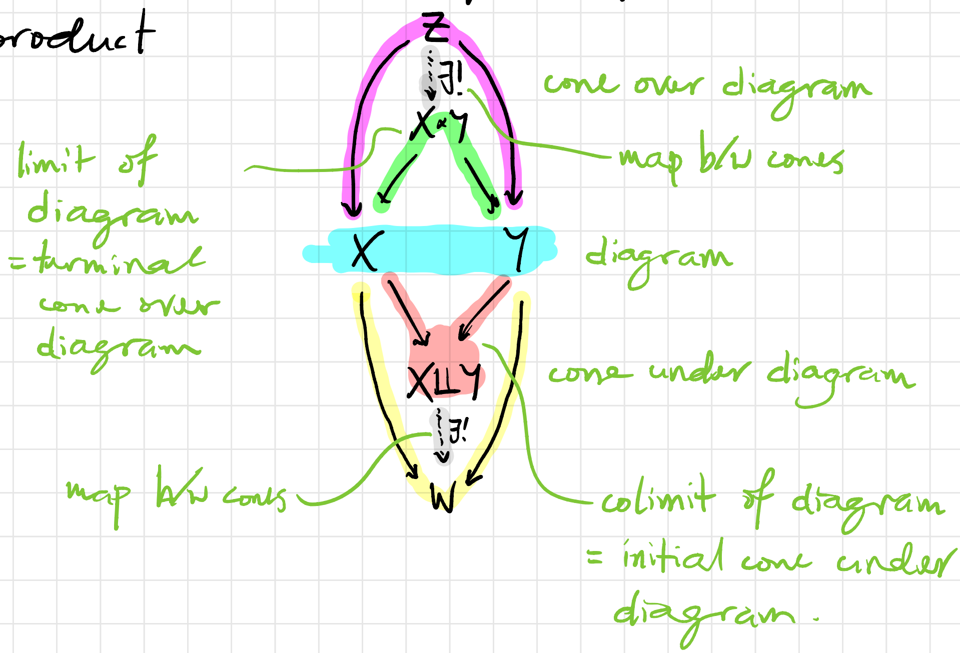
Day 23

Learning Goals

- Diagrams as functors
- Cones over/under a functor or diagram as natural transformations to/from a constant functor.
- Limits/colimits as universal cones over/under a diagram.

Recall the universal properties of product and coproduct



We want to take (co)limits of more general "shapes" which leads us to "diagrams in a category".

Idea A diagram
$$\begin{tikzcd} X \arrow[rr] \arrow[dd] \arrow[dr] & & Y \arrow[dd] \\ & U \arrow[dr] & \\ Z \arrow[rr] & & W \end{tikzcd}$$
in C

is a functor
$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dd] \arrow[dr] & & \bullet \arrow[dd] \\ & \bullet \arrow[dr] & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd} \longrightarrow C$$

small category D
with indicated obj's, morphisms,
diagrams commuting

Defn Let D be a small cat. A *diagram of shape D* in a category C is a functor $D \to C$.

A *morphism* from one diagram to another (necessarily with same shape D, in same cat C) is a natural transformation of functors $D \to C$:

$\eta \in \mathrm{Nat}(F,G)$ for $D = $ $\begin{tikzcd} 1 \arrow[rr] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$ is

$$\begin{tikzcd} F & X \arrow[rr] \arrow[dr] \arrow[dd, "\eta_1"'] & & Z \arrow[dd, "\eta_3"] \\ & & Y \arrow[ur] \arrow[dd, "\eta_2" near start] & \\ G & A \arrow[rr] \arrow[dr] & & C \\ & & B \arrow[ur] & \end{tikzcd}$$

commuting in C.

For any D, C, and $A \in \mathrm{ob}\, C$, there is a **constant diagram**

$$\begin{tikzcd} D \arrow[r, "\mathrm{const}_A"] & C \\ \bullet \arrow[r, mapsto] & A \\ \bullet \arrow[d] \arrow[r, mapsto] & A \arrow[d, "\mathrm{id}_A"] \\ \bullet & A \end{tikzcd}$$

By abuse of notation, we write A for const_A.

<u>Defn</u> Given a diagram $F : D \to C$, a map from A to F (i.e. an elt of $\mathrm{Nat}(\mathrm{const}_A, F)$) is called a **cone over F** (or cone from A to F). A map from F to A is called a **cone under F** (or cone from F to A).

Thus a cone over F is an object A of C and maps $\{\eta_\bullet : A \to F\bullet \mid \bullet \in \mathrm{ob}\, D\}$ such that

$$\begin{tikzcd} & A \arrow[dl, "\eta_\bullet"'] \arrow[dr, "\eta_\circ"] & \\ F\bullet \arrow[rr, "F\varphi"'] & & F\circ \end{tikzcd}$$

commutes $\forall\, \varphi \in D(\bullet, \circ)$.

<u>E.g.</u> A cone over
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z \\ & Y \arrow[u] \end{tikzcd}$$
is
$$\begin{tikzcd} & A \arrow[dl, "\eta_1"'] \arrow[d, "\eta_2"] \arrow[dr, "\eta_3"] & \\ X \arrow[rr] \arrow[dr] & Y & Z \\ & Y \arrow[ur] & \end{tikzcd}$$
that commutes.

A cone under $X \to Z$ is
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z \\ & Y \arrow[u] \end{tikzcd}$$
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] \arrow[ddr, bend right, "\eta_1"'] & & Z \arrow[ddl, bend left, "\eta_3"] \\ & Y \arrow[ur] \arrow[d, "\eta_2"] & \\ & A & \end{tikzcd}$$
that commutes.

Defn A **limit** of the diagram $F: D \to C$ is a cone η from an object $\lim F$ to F which is **terminal** among cones over F: $\forall$ cone $\alpha: \underset{\text{ob } C}{\overset{\cap}{B}} \Rightarrow F$ $\exists!\, h: B \to \lim F$ s.t. $\alpha_\bullet = \eta_\bullet h$ $\forall \bullet \in \text{ob } D$.

$$\begin{tikzcd} & B \arrow[ddl, bend right, "\alpha_\bullet"'] \arrow[d, dashed, "\exists! h"] \\ & \lim F \arrow[dl, "\eta_\bullet"] \\ F_\bullet & \end{tikzcd}$$

In other words, every cone over F *factors through* the cone $\eta: \lim F \Rightarrow F$.

Defn A **colimit** of the diagram $F: D \to C$ is a cone $\varepsilon: F \Rightarrow \text{colim } F \in \text{ob } C$ which is **initial** among cones under F: $\forall$ cone $\alpha: F \Rightarrow B \in \text{ob } C$ $\exists!\, h: \text{colim } F \to B$ s.t. $\alpha_\bullet = h \varepsilon_\bullet$ $\forall \bullet \in \text{ob } D$.

$$\begin{tikzcd} F_\bullet \arrow[r, "\varepsilon_\bullet"] \arrow[dr, bend right, "\alpha_\bullet"'] & \text{colim } F \arrow[d, dashed, "\exists! h"] \\ & B \end{tikzcd}$$

(Co)limits of diagrams might fail to exist!

E.g. Product $= \lim(\bullet \quad \bullet)$, coproduct $= \operatorname{colim}(\bullet \quad \bullet)$

$X \times Y = \lim(X \quad Y)$ $\qquad$ $X \amalg Y = \operatorname{colim}(X \quad Y)$

E.g. Pullback

Take $D = \bullet \to \bullet \leftarrow \bullet$. Diagrams of shape D look like $Y \xrightarrow{g} Z \xleftarrow{f} X$ or

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

When it exists, a limit of such a diagram is called a pullback and is denoted

$X \times_Z Y$ or $X \, {}_f\!\times_g Y$.

more "honest" but less common notation.

(comes with maps to X, Y!)

When $P = X \times_Z Y$ we draw

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

which has the universal property

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

i.e. $\exists! \, Q \to P$ making the diagram commute whenever the outer square commutes.

In $\mathcal{C} = \mathbf{Set}$ or $\mathbf{Top}$, we claim

$$X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

(topologized as a subspace of $X \times Y$ for $\mathcal{C} = \mathbf{Top}$).

This is a cone over $X \xrightarrow{f} Z \xleftarrow{g} Y$ via π_X, π_Y proj'n maps (check!).

If
$$\begin{tikzcd} Q \arrow[r, "i"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
commutes, we may take $h(q) = (i(q), j(q)) \in X \times_Z Y$ and this formula is in fact mandated by commutativity of

$$\begin{tikzcd} Q \arrow[dr, "h"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "j"'] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \\ & Y & \end{tikzcd}$$

so h is unique.

Fact When they exist, (co)limits are unique up to unique isomorphism. We will prove this and see more examples Monday!