Day 22

Learning Goals

- Categories, functors, and natural transformations
- Yoneda lemma

Categories : objects + morphisms

Functors : morphisms between categories

Natural transformations : morphisms between functors!

(The hierarchy continues, leading to "higher category theory" — n-cats, ∞-cats, etc. We'll stop here!)

Recall A functor $F : C \to D$ is an assignment $X \mapsto FX$ of objects of C to objects ~~of~~ D +

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} FX \arrow[d, "Ff"] \\ FY \end{tikzcd}$$

taking morphisms to morphisms such that $F(fg) = (Ff)(Fg)$ and $F(id_X) = id_{FX}$.

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "fg"'] & & Z \end{tikzcd} \longmapsto \begin{tikzcd} & FX \arrow[dr, "Ff"] & \\ FX \arrow[ur, "Fg"] \arrow[rr] & & FZ \end{tikzcd}$$

$F(fg) = Ff\,Fg$

Now suppose we have parallel functors $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & D \end{tikzcd}$.

There is only one reasonable way to relate the data $\begin{matrix} FX \xrightarrow{Ff} FY \\ GX \xrightarrow[Gf]{} GY \end{matrix}$, namely with vertical morphisms

$$\begin{matrix} FX & & FY \\ \eta_X \downarrow & & \downarrow \eta_Y \\ GX & & GY \end{matrix}$$

such that the diagram

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\eta_X"'] & FY \arrow[d, "\eta_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

commutes $\forall f$. Formally:

<u>**Defn**</u> A *natural transformation* η from $F: C \longrightarrow D$ to $G: C \longrightarrow D$ consists of $\eta_X : FX \longrightarrow GX$ a morphism in D for each $X \in \text{ob}\, C$ s.t. $\forall f \in C(X, Y)$,

$\eta_Y \, Ff = Gf \, \eta_X$ (i.e., the above diagram commutes).

We write $\eta : F \Rightarrow G$, and we let $\text{Nat}(F, G)$ denote all natural transformations from F to G.

If η_X is an isomorphism $\forall X \in \text{ob}\, C$, then we call η a *natural isomorphism*.

More notation:

$$\begin{tikzcd} C \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & D \arrow[from=U, to=D, phantom, "\Downarrow\eta"] \end{tikzcd}$$

(something 2-dimensional about this...)

E.g. There are functors $\mathcal{O}, \mathcal{C} : \mathbf{Top}^{op} \longrightarrow \mathbf{Set}$

with $\mathcal{O}(X) = \{\text{open sets of } X\}$, $\mathcal{C}(X) = \{\text{closed sets of } X\}$

and $\underset{\substack{\| \\ f^*}}{Ff} : \underset{\substack{\cap \\ FY}}{S} \longmapsto \underset{\substack{\cap \\ FX}}{f^{-1}S}$ for $f \in \mathbf{Top}(X, Y)$ and $F = \mathcal{O}$ or $\mathcal{C}$.

Define $\eta_X : \mathcal{O}(X) \xrightarrow{\cong} \mathcal{C}(X)$
$\qquad\qquad U \longmapsto X \setminus U$. Then

$$\begin{tikzcd} \mathcal{O}(Y) \arrow[r, "f^*"] \arrow[d, "\eta_Y"'] & \mathcal{O}(X) \arrow[d, "\eta_X"] \\ \mathcal{C}(Y) \arrow[r, "f^*"'] & \mathcal{C}(X) \end{tikzcd}$$

commutes so η is a natural isomorphism between open and closed sets functors.

E.g. For sets A, B, C,

$$A \times (B \amalg C) \cong (A \times B) \amalg (A \times C)$$

$$(A \times B)^C \cong A^C \times B^C$$

$$A^{B \amalg C} \cong A^B \times A^C$$

$$(A^B)^C \cong A^{B \times C}$$

and all these isos are natural (exc!).

Restricting to finite sets and taking cardinalities, we get the familiar relationships between addition, multiplication, and exponentiation. Thus FinSet_{iso} is a categorification of arithmetic on $\mathbb{N}$

via $|\ |: \mathrm{FinSet}_{iso} \longrightarrow \mathbb{N}$.

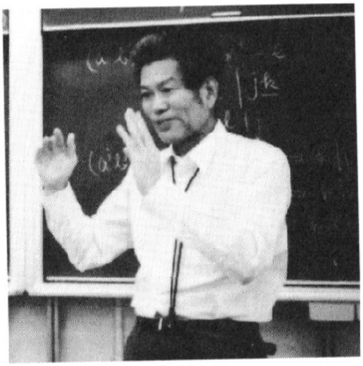
cardinality

ob $\mathbb{N}$
identity morphisms only.

Recall For $X \in \mathrm{ob}\, C$ have the representable functor

$$C(-,X): C^{op} \longrightarrow \mathrm{Set}$$
$$Y \longmapsto C(Y,X)$$

$$\begin{tikzcd} Y \arrow[d, "f"'] & \\ Z & \end{tikzcd} \ \in C \longmapsto \begin{tikzcd} C(Z,X) \arrow[d] & g \arrow[d, mapsto] \\ C(Y,X) & gf \end{tikzcd}$$



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Nobuo Yoneda (1930 - 96)

Yoneda Lemma For $X \in \mathrm{ob}\, C$ and $F: C^{op} \longrightarrow \mathrm{Set}$ a functor, there is a bijection

$$\Phi: \mathrm{Nat}(C(-,X), F) \xrightarrow{\cong} FX$$
$$\eta \longmapsto \eta_X(\mathrm{id}_X)$$

This bijection is natural in X and F.

Pf of bijectivity. We construct an inverse function

$$\Psi : FX \longrightarrow \mathrm{Nat}(C(-,X), F)$$

$$\begin{tikzcd}
x \arrow[r, mapsto] & C(Y,X) \arrow[d, "\Psi(x)_Y"] & f \arrow[d, mapsto] \\
& FY & Ff(x)
\end{tikzcd}.$$

For naturality of $\Psi(x)$, given $f \in C(Y,Z)$ consider the diagram

$$\begin{tikzcd}
Z \xrightarrow{g} X \arrow[rrr, mapsto, bend left] \arrow[ddd, mapsto] & & & Y \xrightarrow{gf} X \ \left(Y \xrightarrow{f} Z \xrightarrow{g} X\right) \arrow[dd, mapsto, "\left(\text{for } FY \xleftarrow{F(gf)} FX \ni x\right)"] \\
& C(Z,X) \arrow[r, "f^*"] \arrow[d, "\Psi(x)_Z"'] & C(Y,X) \arrow[d, "\Psi(x)_Y"] & \\
& FZ \arrow[r, "Ff"'] & FY & (F(gf))(x) \arrow[d, equal] \\
Fg(x) \arrow[rrr, mapsto, bend right] & & & Ff(Fg(x))
\end{tikzcd}$$

(for $FZ \xleftarrow{Fg} FX \ni x$)

(equal since $F : C^{op} \to \mathrm{Set}$ a functor: $F(gf) = (Ff)(Fg)$)

which commutes.

Remains to show Φ and Ψ are inverse.

We have $\Phi\Psi(x) = \Psi(x)_X(id_X) = F\,id_X(x) = id_{FX}(x) = x$.

Moreover, $\Psi\Phi(\eta)_Y = \Psi(\eta_X(id_X))_Y : f \longmapsto Ff(\eta_X(id_X))$.
(for $f \in C(Y,X)$)

By naturality of η, the square

$$\begin{tikzcd} C(X,X) \arrow[r, "f^*"] \arrow[d, "\eta_X"'] & C(Y,X) \arrow[d, "\eta_Y"] \\ FX \arrow[r, "Ff"'] & FY \end{tikzcd}$$

commutes, so $\Psi\Phi(\eta)_Y(f) = \eta_Y(f^* id_X) = \eta_Y(f)$

$\forall f \in C(Y,X)$, i.e. $\Psi\Phi(\eta) = \eta$.

This proves Φ, Ψ are inverse bijections. $\square$

Pf of Naturality Moral exc / cf. Riehl pp. 58-59. $\square$

We get the Yoneda embedding as a corollary of naturality:

$$\begin{aligned} y : C &\longrightarrow Set^{C^{op}} \\ X &\longmapsto C(-,X) \\ f\downarrow &\longmapsto \downarrow f_* \\ Y &\longmapsto C(-,Y) \end{aligned}$$

category of functors $C^{op} \to Set$ + natural transf'ns.

a full and faithful embedding of categories!

See Riehl pp. 60-61 for how to deduce the following from Yoneda:

- every row operation on a matrix arises from left multiplication by an elementary matrix
- Cayley's Thm Any group G is isomorphic to a subgroup of $\mathfrak{S}_{|G|}$.