Day 21

Learning goals

- Tychonoff's Thm: Arbitrary product of compact spaces is compact.

Recall Opens in $\prod_{\alpha \in A} X_\alpha$ are unions of finite int'ns of sets of the form $\pi_\alpha^{-1} U$, $U \subseteq X_\alpha$ open.

Pf Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of compact spaces.

Define $X := \prod_{\alpha \in A} X_\alpha$ with product top.

Let $\mathcal{F}$ be an ultrafilter on X. WTS $\mathcal{F}$ converges.

Since pushforwards of uf's along cts maps are uf's, have uf $(\pi_\alpha)_* \mathcal{F}$ on X_α $\forall \alpha \in A$. Since each X_α is compact, $\exists x_\alpha \in X_\alpha$ s.t. $(\pi_\alpha)_* \mathcal{F} \longrightarrow x_\alpha \in X_\alpha$.

Thus $\forall U \in \mathcal{T}_{x_\alpha}$ $\exists B \in \mathcal{F}$ s.t. $\pi_\alpha B \subseteq U$.

Equiv, $B \subseteq \pi_\alpha^{-1} U \Rightarrow \pi_\alpha^{-1} U \in \mathcal{F}$. Every open nbhd of $(x_\alpha)_{\alpha \in A} \in X$ is a union of finite int'ns of the $\pi_\alpha^{-1} U \Rightarrow \mathcal{T}_{(x_\alpha)_{\alpha \in A}} \subseteq \mathcal{F} \Rightarrow \mathcal{F} \to (x_\alpha)_{\alpha \in A}$.

$\square$

That's it! UF's gave us a "simple" proof since they (a) detect compactness

(b) play well with cts fns.

We could have tried a similar approach w/sequences, but Bolzano-Weierstrass doesn't have a converse in general.

This is a typical approach in mathematics. Given a "reasonable approach" to a problem, either

(a) add hypotheses until it works, or

(b) define/construct new tools that generalize the original concept but apply in more settings.

Or (c) keep grinding!

Approach (b) is nicely conveyed by Grothendieck's "Rising Sea" metaphor:

The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it.. yet it finally surrounds the resistant substance.