# Day 20

## Learning goals

- Ultrafilters = maximal proper filters
- Ultrafilters = prime filters
- Zorn's lemma $\Longrightarrow$ every proper filter $\subseteq$ ultrafilter
- Compact = every prime filter converges

Our ultimate goal with filters is to give an elegant proof of Tychonoff's Thm: every product of compact spaces is compact. To do so, we need a characterization of compactness in terms of ultrafilters.

**Defn** A proper filter on a set is an *ultrafilter* when it is not properly contained in any other proper filter.

I.e. ultrafilters = maximal proper filters.

**Prop** A filter $\mathcal{F}$ on $X$ is an ultrafilter iff $\forall A \subseteq X$, $A \notin \mathcal{F}$ iff $\exists B \in \mathcal{F}$ with $A \cap B = \emptyset$.

**Pf** ($\Rightarrow$) Let $\mathcal{F}$ be an ultrafilter. Then $A \notin \mathcal{F}$ iff the filter gen'd by $\mathcal{F} \cup \{A\}$ is $2^X$.

**Claim** The filter gen'd by $\mathcal{F} \cup \{A\}$ is
$$\{ C \subseteq X \mid B \cap A \subseteq C \text{ for some } B \in \mathcal{F} \}.$$
(True since this is the upward closure of the closure of $\mathcal{F} \cup \{A\}$ under finite int'n.)

Thus $A \notin \mathcal{F}$ iff $B \cap A = \emptyset$ for some $B \in \mathcal{F}$, as desired.

($\Leftarrow$) Suppose $\mathcal{F}$ satisfies the condition, and suppose $\mathcal{F} \subsetneq \mathcal{G}$, $\mathcal{G}$ some filter. WTS $\mathcal{G} = 2^X$.

Know $\exists A \in \mathcal{G}$ s.t. $A \notin \mathcal{F}$. Thus $\exists B \in \mathcal{F}$ s.t. $B \cap A = \emptyset$.

But then $\emptyset \in \mathcal{G}$ so $\mathcal{G} = 2^X$. $\square$

**E.g.** For any $x \in X$, the principal filter at $x$ is $\mathcal{P}(x) = \{ A \subseteq X \mid x \in A \}$. This is an ultrafilter:

$A \notin \mathcal{P}(x) \iff x \notin A$ so $\{x\} \cap A = \emptyset$ and $\{x\} \in \mathcal{P}(x)$ so $\mathcal{P}(x)$ is principal.

**Defn** A filter $\mathcal{F}$ on $X$ is prime iff $\mathcal{F}$ is proper and $\forall A, B \subseteq X$,
$$A \cup B \in \mathcal{F} \implies A \in \mathcal{F} \text{ or } B \in \mathcal{F}.$$

**Thm** A filter on $X$ is maximal iff it is prime.

**NB** For filters in posets other than $(2^X, \subseteq)$, maximality and primality are distinct!

by way of

**Pf** ($\Rightarrow$) Suppose $\mathcal{F}$ an ultrafilter and BWOC assume $\mathcal{F}$ not prime. Take $A, B \subseteq X$ with $A \cup B \in \mathcal{F}$ but $A, B \notin \mathcal{F}$. Then $\exists A', B' \in \mathcal{F}$ with $A \cap A' = \emptyset = B \cap B'$. Furthermore, $A' \cap B' \in \mathcal{F}$ and $(A \cup B) \cap (A' \cap B') = \emptyset$ $\Rightarrow A \cup B \notin \mathcal{F}$, contradicting $A \cup B \in \mathcal{F}$.

Typo in book's proof.

($\Leftarrow$) Suppose $\mathcal{F}$ prime and BWOC assume $\mathcal{F}$ not maximal. Then $\mathcal{F} \subsetneq \mathcal{G} \subsetneq 2^X$, $\mathcal{G}$ a filter. Take $\emptyset \neq A \in \mathcal{G}$ with $A \notin \mathcal{F}$. If $X \setminus A \in \mathcal{F}$ then $X \setminus A \in \mathcal{G} \Rightarrow$ $\emptyset = A \cap (X \setminus A) \in \mathcal{G}$, contradicting properness. Thus $X \setminus A \notin \mathcal{F}$. But now $X = A \cup (X \setminus A) \in \mathcal{F}$ with $A, X \setminus A \notin \mathcal{F}$ contradicting primality of $\mathcal{F}$. $\square$

**NB** We can now interchange maximality and primality for ultrafilters in proofs.

**Zorn's lemma** If every chain in a nonempty poset $P$ has an upper bound, then $P$ has a maximal elt.
(Equivalent to the axiom of choice!)

**Ultrafilter lemma** Every proper filter is contained in an ultrafilter.

Pf Any set of filters $\{\mathcal{F}_\alpha \mid \alpha \in A\}$ is bounded above by the upward closure of the filterbase of finite int'ns of elts of the $\mathcal{F}_\alpha$. When $\{\mathcal{F}_\alpha \mid \alpha \in A\}$ is a chain of proper filters, this upper bound is itself proper (check this ☺). Thus chains of proper filters containing a fixed proper filter have upper bounds. By Zorn, $\exists$ max'l filter containing $\mathcal{F}$. $\square$

not necessarily unique! All ultrafilters contain $\{x\}$.

## Other Zorny Facts

- All vector spaces have bases
- Every ring has maximal ideals.

Cor Every infinite set has a non-principal ultrafilter.

Pf Consider the Fréchet filter $\mathcal{F} = \{A \subseteq X \mid X \setminus A \text{ finite}\}$. Extend to an ultrafilter $\mathcal{U}$. If $\mathcal{U} = \mathcal{P}(x)$, then $\{x\} \in \mathcal{U}$. But then $X \setminus \{x\} \in \mathcal{F} \subseteq \mathcal{U}$ so $\emptyset = \{x\} \cap (X \setminus \{x\}) \in \mathcal{U}$. So $\mathcal{U} = 2^X$ ☠. $\square$

maximal!

Thm $X$ is compact iff every prime filter converges.

Converse to Bolzano-Weierstrass:
$X$ cpt $\Rightarrow$ every sequence has a convergent subsequence

Here subsequence $\rightsquigarrow$ ultrafilter containing the eventuality filter of the sequence.

Pf Suppose $\mathcal{F}$ prime, $\mathcal{F} \not\to x$ $\forall x \in X$. Then $\forall x \in X$ $\exists U_x \in \mathcal{T}_x \setminus \mathcal{F}$. The set $\{U_x \mid x \in X\}$ is an open cover. By compactness, choose finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Then $U_{x_1} \cup \dots \cup U_{x_n} = X \in \mathcal{F}$. By primality some $U_{x_i} \in \mathcal{F}$ ↯.

Now suppose $X$ not cpt. Choose $\mathcal{V}$ collection of closed sets with FIP and empty int'n.
Then $\forall x \in X$ $\exists V_x \in \mathcal{V}$ with $x \notin V_x$. By ultrafilter lemma, $\mathcal{V}$ is contained in a ultrafilter $\mathcal{U}$. But $\mathcal{U} \not\to x$ for any $x$ b/c o/w $V_x \cap V_x^c = \emptyset \in \mathcal{U}$ and $\mathcal{U}$ improper ↯ $\square$

FIP: finite int'n property every finite int'n $\neq \emptyset$

Thm $X$ cpt iff every collection closed subsets of $X$ with FIP has nonempty int'n

Cor A space is compact Hausdorff iff every prime filter converges to exactly 1 pt. $\square$

Thm $\mathcal{U}$ an ultrafilter on $X$, fn $f: X \to Y$ $\Rightarrow$ $f_*\mathcal{U}$ an ultrafilter on $Y$. $\square$

Thus get a functor

$$\beta : \mathrm{Set} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \beta X := \{\text{ultrafilters on } X\}$$

$$\begin{array}{ccccc} X & & \beta X & & \mathcal{U} \\ f\downarrow & \longmapsto & \downarrow \beta f & & \downarrow \\ Y & & \beta Y & & f_*\mathcal{U} \end{array}$$

For $X$ cpt H'f, the map $\alpha : \beta X \longrightarrow X$, $\mathcal{U} \longmapsto \lim \mathcal{U}$ plays a distinguished role. See text for more. We'll see this again when we study Stone-Čech compactification.