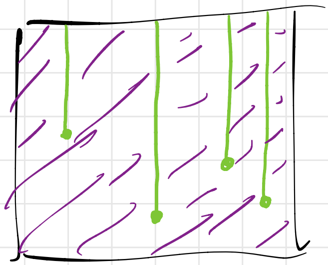
Day 19

Learning Goals

- Define filters
  - examples
  - convergence
- Hausdorff spaces, closed sets, and continuity are detected by filters.

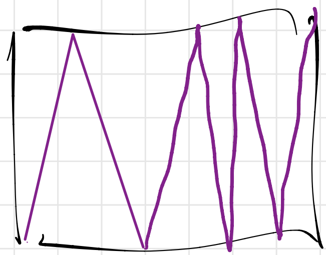
Motivating Example

Let $X = [0,1]^{[0,1]}$, the set of functions $[0,1] \to [0,1]$ with the product topology. Open nbhds of the zero function $z(x) = 0 \ \forall x \in [0,1]$ have half open intervals "in finitely many coords" and $[0,1]$ in others:



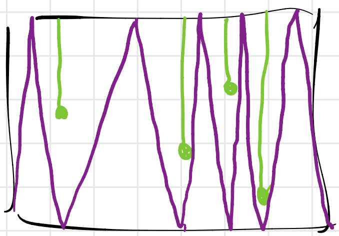
open

Let $A \subseteq X$ be the set of "sawtooth functions"



with finitely many height 1 spikes (exclude z).

Given a nbhd of z, we can find a pt in A not equal to z inside the nbhd:



Thus z is a limit point of $A \Rightarrow z \in \bar{A}$.

But no sequence in A converges to z:

If $(f_n) \to z$ then $\forall x \in [0,1]$, $(f_n(x)) \to 0$

but fns in A all take value 1 somewhere!

$\circ \circ \circ$ { When X is not first countable, need to probe w/ something more refined than sequences.

Defn A *filter* on a set X is $\mathcal{F} \subseteq 2^X$ that is

(1) *downward directed*: $A, B \in \mathcal{F} \Rightarrow \exists C \in \mathcal{F}$ s.t. $C \subseteq A \cap B$

(2) *nonempty*: $\mathcal{F} \neq \emptyset$

(3) *upward closed*: $A \in \mathcal{F}$, $A \subseteq B \Rightarrow B \in \mathcal{F}$.

A filter is *proper* when $\mathcal{F} \neq 2^X$. ← the improper filter

TPS Show that filters are closed under finite intersections.

Let $2 = \{0 \leq 1\}$ as a poset and define $i \wedge j = \inf\{i, j\}$ ← meet

Note that for $A, B \in 2^X$, $A \wedge B = A \cap B$.

Prop Suppose $f : 2^X \longrightarrow 2$ is a monotone function ($A \subseteq B \Rightarrow f(A) \leq f(B)$) preserving meets ($f(A \wedge B) = f(A) \wedge f(B)$). Then $f^{-1}\{1\}$ is a filter. Moreover, the assignment

$$\left\{ f : 2^X \longrightarrow 2 \;\middle|\; \begin{array}{l} \text{monotone,} \\ \text{meet-preserving} \end{array} \right\} \longrightarrow \{\text{filters on } X\}$$

$$f \longmapsto f^{-1}\{1\}$$

is a bij'n. $\square$

E.g. (1) $\mathcal{F} = \{X\}$ is called the trivial filter.

(2) For (x_n) a sequence in X, its eventuality filter is $\mathcal{E}_{(x_n)} = \{A \subseteq X \mid x_n \in A \text{ for } n \gg 0\}$.

(3) The Fréchet filter on X is $\{A \subseteq X \mid X \setminus A \text{ finite}\}$.

Non-e.g. For $x \in X$, let $\mathcal{T}_x = \{U \subseteq X \mid x \in U \text{ open}\}$. This is a nbhd base of x and it is downward directed and nonempty, but not (in gen'l) upward closed.
(This makes $\mathcal{T}_x$ a filterbase and its upward closure is a filter.)

Defn A filter $\mathcal{F}$ on a space X converges to $x \in X$ when $\mathcal{T}_x \subseteq \mathcal{F}$; write $\mathcal{F} \to x$.

Prop A sequence $(x_n) \to x$ iff $\mathcal{E}_{(x_n)} \to x$.

Pf $\mathcal{T}_x \subseteq \mathcal{E}_{(x_n)} \iff \forall\, U \ni x$ open, $x_n \in U$ for $n \gg 0$. $\square$

E.g. $2^X \to x$ $\forall x \in X$ — thus we will only care about proper filters!

Motivating example, ct'd $A \subseteq [0,1]^{[0,1]}_{\text{prod}}$ consisting of sawtooth functions. Let $\mathcal{B} = \{U \cap A \mid U \in \mathcal{T}_z\}$. This is a filterbase generating a filter $\mathcal{F}$ (the upward closure of $\mathcal{B}$) which contains $A = X \cap A$. Also $U \cap A \neq \emptyset$ $\forall U \in \mathcal{T}_z$ b/c z is a limit point of A. It follows that $\mathcal{F}$ is proper (in fact a filter is proper iff $\emptyset \notin \mathcal{F}$). We have $\mathcal{T}_z \subseteq \mathcal{F}$ since $U \cap A \subseteq U$ $\forall U \in \mathcal{T}_z$. Thus $\mathcal{F}$ is a proper filter containing A converging to z.

Thm For $A \subseteq X$ a subset of a space, $x \in \bar{A}$ iff $\exists$ proper filter $\mathcal{F}$ on X s.t. $A \in \mathcal{F}$ and $\mathcal{F} \to x$.

Pf ($\Leftarrow$) Follow the motivating example.

($\Rightarrow$) If $\mathcal{F}$ is a proper filter, $\mathcal{F} \to x$, $A \in \mathcal{F}$, then $\mathcal{B} = \{U \cap A \mid U \in \mathcal{T}_z\} \subseteq \mathcal{F} \Rightarrow \emptyset \notin \mathcal{B} \Rightarrow x \in \bar{A}$.

$\square$

Thm X is Hausdorff $\Leftrightarrow$ limits of convergent proper filters are unique.

Pf See 3.8. $\square$

To examine continuity with filters, we need a defn:

Defn Given a filter $\mathcal{F}$ on X and function $f: X \to Y$, the set $\{fA \mid A \in \mathcal{F}\}$ is a filterbase on Y. The filter $f_*\mathcal{F}$ generated by this base is the **pushforward of $\mathcal{F}$ wrt f**. Explicitly,

$$f_*\mathcal{F} = \{B \subseteq Y \mid \exists A \in \mathcal{F} \text{ s.t. } fA \subseteq B\}.$$

E.g. $f_* \mathcal{E}_{(x_n)} = \mathcal{E}_{(f(x_n))}$.

Thm A function $f: X \to Y$ is cts iff $\forall$ filter $\mathcal{F}$ on X, if $\mathcal{F} \to x$ then $f_*\mathcal{F} \to f(x)$.

Pf ($\Rightarrow$) Suppose f cts, $\mathcal{F} \to x$. WTS $\mathcal{T}_{f(x)} \subseteq f_*\mathcal{F}$ i.e. for $B \in \mathcal{T}_{f(x)}$ $\exists A \in \mathcal{F}$ with $fA \subseteq B$.

Choose $A = f^{-1}B$. By continuity, $f^{-1}\mathcal{T}_{f(x)} \subseteq \mathcal{T}_x$ so $A \in \mathcal{T}_x$. Know $\mathcal{T}_x \subseteq \mathcal{F}$ so $A \in \mathcal{F}$.

($\Leftarrow$) Suppose $\forall \mathcal{F}$, $\mathcal{F} \to x \Rightarrow f_*\mathcal{F} \to f(x)$. Let $\mathcal{F}$ be the filter gen'd by $\mathcal{T}_x$. Then $\mathcal{F} \to x$ so $f_*\mathcal{F} \to f(x)$, so $\mathcal{T}_{f(x)} \subseteq f_*\mathcal{F}$. Thus $\forall B \in \mathcal{T}_{f(x)}$, $\exists A \in \mathcal{T}_x$ s.t. $fA \subseteq B$. This condition is equivalent to continuity of f. $\square$