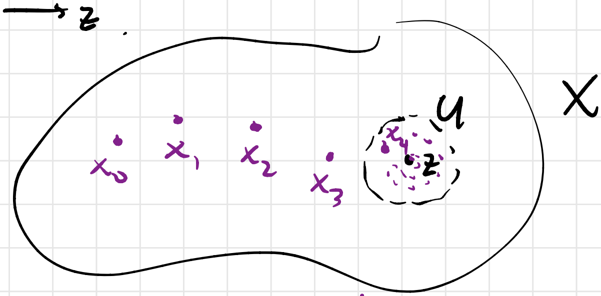
Day 18

Learning goals

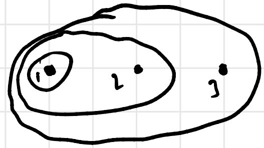
- Sequences in top spaces
- $T_1 \iff$ constant sequence (x) converges to x and only x
- Hausdorff $\Rightarrow$ at most one limit
- sequences in A converge to points in $\bar{A}$
- $f : X \to Y$ cts, $(x_n) \to x \in X \Rightarrow (f(x_n)) \to f(x) \in Y$
- first and second countability

Defn A *sequence* in X is a function $x : \mathbb{N} \longrightarrow X$, $n \longmapsto x_n$

Write $x = (x_n)$. A sequence (x_n) *converges* to $z \in X$ when $\forall\, U \subseteq X$ open containing z, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n \in U$. When (x_n) converges to z, write $(x_n) \longrightarrow z$.



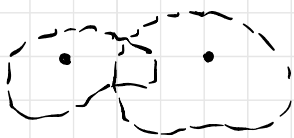
$N = 4$

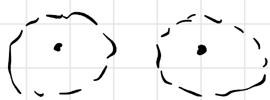
Eg. (a)  $(1) \to 1, 2, 3$

(b) $\mathbb{Z}_{cofin}$, $(m) \to m$ and only m: $\mathbb{R}\setminus\{m\}$ open nbhd of $l \neq m$ containing no pts of the sequence

(b′) In $\mathbb{Z}_{cofin}$, $(n) = (0, 1, 2, \dots) \to m$ $\forall m \in \mathbb{Z}$:
take U nbhd of m, $\mathbb{Z}\setminus U$ finite so for $N = \max(\mathbb{Z}\setminus U) + 1$, have $n \in U$ for $n \geq N$.

Recall T_0: 

T_1:  $\iff$ pts closed

T_2:  Hausdorff

Thm X is T_1 $\iff$ $\forall x \in X$, $(x) \to x$ and only x.

Pf $(\Rightarrow)$ $(x) \to x$ ✓
Let $y \neq x$. Then $\exists\, U \ni y$ open, $x \notin U$, so $(x) \not\to y$.

$(\Leftarrow)$ Suppose X not T_1. $\exists\, x \neq y \in X$ s.t. $y \in \underset{\text{open}}{U} \Rightarrow x \in U$

so $(x) \to y$. $\square$

Thm X Hausdorff $\Rightarrow$ seq's in X have at most one limit.

Pf Suppose X Hausdorff and $(x_n) \to x$, $(x_n) \to y$

If $x \neq y$, take disjoint open $\underset{x}{\overset{U}{\cup}}, \underset{y}{\overset{V}{\cup}}$

$\exists N$ s.t. $x_n \in U$ for $n \geq N$

$\exists K$ s.t. $x_n \in V$ for $n \geq K$.

For $n \geq \max\{N, K\}$, $x_n \in U \cap V$ $\emptyset$. $\square$

Thm (x_n) a sequence in $A \subseteq X$, $(x_n) \to x$ then $x \in \overline{A}$.

Recall $\overline{A} = \bigcap_{C \supseteq A \text{ closed}} C$ = smallest closed set containing A

$= A \cup \{\text{limit pts of } A\}$.

moral exc

every open containing x contains a point of $A - \{x\}$.

Pf Thm If (x_n) is eventually constant at x, then $x \in A$. Assume (x_n) is not constant at x for $n >> 0$. If $U \ni x$ open, then $U \ni x_n$ for $n >> 0$ $\Rightarrow U$ contains points of A other than

x for $n \gg 0$. $\square$

Thm $f: X \to Y$ cts, $(x_n) \to x$ in X then $(f(x_n)) \to f(x)$ in Y.

Pf If $f(x) \in U \subseteq Y$ open, then $x \in f^{-1}U \subseteq X$ open. For $n \gg 0$, $x_n \in f^{-1}U \Rightarrow f(x_n) \in f(f^{-1}U) \subseteq U$, $\square$

⚠ Converses are false: seq's do not detect Hausdorffness, closed sets, or continuity in gen'l.

Read Ex 3.4, 3.5, 3.6 for specific examples.

They involve spaces like $\mathbb{R}_{\text{cocountable}}$, $[0,1]^{[0,1]}$ with "lots" of opens around each point.

Controlling # opens around each point might make sequences sufficient to probe these properties.

Defn X a space. A collection of open sets $\mathcal{B}$ is a *neighborhood base* for $x \in X$ when $\forall$ open $O \ni x$, $\exists U \in \mathcal{B}$ with $x \in U \subseteq O$. A space is called *first countable* when every point has a countable nbhd base.

A space is called **second countable** when it has a countable basis.

Eg. Every metric space is first countable with $\{B(x,1), B(x,\frac{1}{2}), B(x,\frac{1}{3}), \dots\}$ forming a nbhd base of x.

Thm For X first countable, X is Hausdorff $\Leftrightarrow$ every sequence has ≤ 1 limit.

Pf Just need to check $\Leftarrow$. For contraposition, suppose X is not Hausdorff and take x, y not seperated by open sets. Take $U_1, U_2, \dots$ nbhd base of x, $V_1, V_2, \dots$ nbhd base of y. Replacing U_n with $U_1 \cap \dots \cap U_n$ and sim for V_n, we may assume $U_1 \supseteq U_2 \supseteq \dots$ and $V_1 \supseteq V_2 \supseteq \dots$.

Now for each n take $x_n \in U_n \cap V_n \neq \emptyset$.

Claim $(x_n) \to x$ and y.

If $U \ni x$ open, then $\exists U_N \subseteq U$ and $U_n \subseteq U_N$ for $n \geq N$. Thus

$$\underset{x_{N+2}}{\overset{\cdots}{\scriptstyle\cup}} \subseteq \underset{x_{N+1}}{U_{N+1}} \subseteq \underset{x_N}{U_N} \subseteq \underset{x}{U}$$

so $x_n \in U$ for $n \geq N$.

Sim for $V \ni y$ open. $\square$

Thm For X first countable, $x \in \bar{A} \iff \exists (x_n) \to x$, $x_n \in A$.

Thm For X, Y first countable, $f: X \to Y$ cts $\iff \left[(x_n) \to x \in X \implies (f(x_n)) \to f(x) \in Y \right]$.

We will use filters instead of sequences to detect these properties for non-first countable spaces.