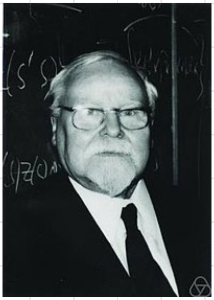
Day 17

Learning Goals

- Tychonoff $\Rightarrow$ Heine-Borel
- Extreme value theorem
- Tube lemma
- Uncountability of nonempty cpt Hausdorff spaces w/o isolated pts

Thm [Tychonoff] Arbitrary products of compact spaces are compact.



Tychonoff
1906-93

Pf via ultrafilters in 2-3 weeks.

Cor [Heine-Borel Thm] A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Pf Suppose $K \subseteq \mathbb{R}^n$ is compact. The open cover $\mathcal{U} = \{B(0,r) \mid r > 0\}$ has a finite subcover, so K is bounded. Since $\mathbb{R}^n$ is Hausdorff, K is also closed.

Now suppose $K \subseteq \mathbb{R}^n$ is closed and bounded. Then by boundedness, $K \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n]$ for some a_i, b_i. By Tychonoff, K is a closed subset of a compact set, so K is compact. $\square$

But wait... do we know $[a,b] \subseteq \mathbb{R}$ is compact?

Let $\mathcal{U}$ be an open cover of $[a,b]$ and set $s = \sup C$ for $C = \{c \in [a,b] \mid \mathcal{U} \text{ has a finite subcover for } [a,c]\}$.

Clearly $a \in C$ so $s \in [a,b]$.

Take $U \in \mathcal{U}$ with $s \in U$. Pick $\varepsilon > 0$ s.t. $[s-\varepsilon, s+\varepsilon] \subseteq U$.

By the defn of s, we may take $c \in C$ with $s - \varepsilon < c$.

Thus there is a finite subcover $\mathcal{U}' \subseteq \mathcal{U}$ of $[a,c]$.

Then $\mathcal{U}' \cup \{U\}$ is a finite subcover of $[a,c']$ for $c' = \min\{b, s+\varepsilon\}$. Check that $\mathcal{U}' \cup \{U\}$ is a finite subcover of $[a,b]$. $\square$

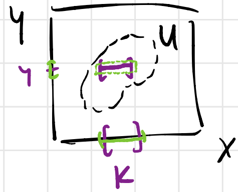
Cor [Extreme Value Thm] If K is a compact space and $f : K \longrightarrow \mathbb{R}$ is cts, then f attains a global max and min.

Pf $fK \subseteq \mathbb{R}$ is compact and hence closed and bounded.

Let $s = \sup fK < \infty$. Suppose for ⨳ that $s \notin fK$.

Then s is in the open set $\mathbb{R} \setminus fK$ so $\exists \varepsilon > 0$ s.t. $(s-\varepsilon, s+\varepsilon) \subseteq \mathbb{R} \setminus fK$. But then $s - \varepsilon$ is a smaller upper bound of fK 오. Sim for $\inf fK$. $\square$

Setup for tube lemma:



Lemma [tube] For any $U \subseteq X \times Y$, $K \subseteq X$ compact, $y \in Y$ $\exists$ open $V \subseteq X$, $W \subseteq Y$ with $K \times \{y\} \subseteq V \times W \subseteq U$.

Pf $\forall (x,y) \in K \times \{y\}$ $\exists$ open $V_x \subseteq X$, $W_x \subseteq Y$ with $(x,y) \in V_x \times W_x \subseteq U$. Then $\{V_x \mid x \in K\}$ is an open cover of K which thus has a finite subcover $\{V_1, \dots, V_n\}$. Then $V = V_1 \cup \cdots \cup V_n$, $W = W_1 \cap \cdots \cap W_n$ work. $\square$

(See Munkres Thm 26.7 for how to use the tube lemma to prove the finite product version of Tychonoff's thm.)

Defn $\mathcal{C} \subseteq 2^X$ has the finite intersection property (or FIP) when $\forall$ finite subcollections $\{C_1, \dots, C_n\} \subseteq \mathcal{C}$, $C_1 \cap \cdots \cap C_n \neq \emptyset$.

Thm X is cpt $\iff$ all collections $\mathcal{C}$ of closed sets in X satisfying FIP have $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Cor If $C_1 \supseteq C_2 \supseteq \cdots$ is a nested sequence of closed nonempty subsets of compact! X, then $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Pf of Thm Given $\mathcal{A} \subseteq 2^X$, let $\mathcal{C}_{\mathcal{A}} := \{X \setminus A \mid A \in \mathcal{A}\}$.

Then (1) $\mathcal{A}$ consists of opens iff $\mathcal{C}_{\mathcal{A}}$ consists of closed sets.

(2) $\mathcal{A}$ covers X iff $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

(3) $\{A_1, \dots, A_n\} \subseteq \mathcal{A}$ covers X iff $C_1 \cap \cdots \cap C_n = \emptyset$ for $C_i = X \setminus A_i$.

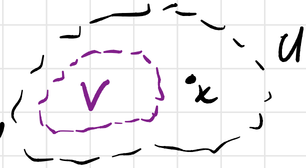
Moral exc Finish the proof by taking the contrapositive of the statement and complements of sets! $\square$

Defn Call $x \in X$ *isolated* when $\{x\} \subseteq X$ is open.

Thm If X is a nonempty compact Hausdorff space with no isolated points, then X is uncountable.

Pf Step 1 Show that $\forall\, \emptyset \neq U \subseteq X$ open and any point $x \in X$ $\exists\, \emptyset \neq V \subseteq X$ open with $V \subseteq U$ and $x \notin \overline{V}$.

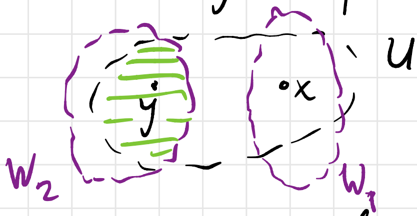
Take $y \in U$, $y \neq x$:



if $x \in U$, x is not isolated so $U \neq \{x\}$; if $x \notin U$, $U \neq \emptyset$.

Use H'ness to take W_1, W_2 disjoint open nbhds of x, y.

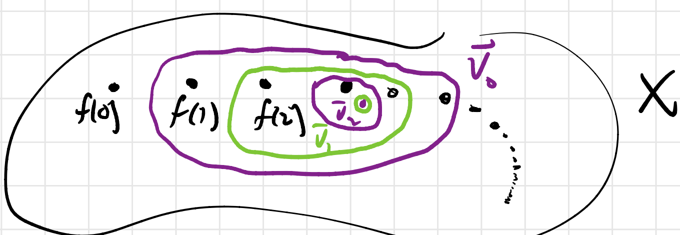
$V := W_2 \cap U$ works.



Step 2 Show that any fn $f : \mathbb{N} \longrightarrow X$ is not surjective (so X is uncountable)

Apply Step 1 to $U = X$ and $x = f(0)$ to produce $V_0 \subseteq X$ with $\overline{V_0} \not\ni f(0)$. Given V_{n-1} open nonempty, choose $V_n \neq \emptyset$ open with $V_n \subseteq V_{n-1}$ and $\overline{V_n} \not\ni f(n)$.

Get the nested sequence $\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$ of nonempty closed sets. Since X is cpt, $\exists x \in \bigcap \overline{V_n}$ and $x \neq f(n)$ $\forall n \in \mathbb{N}$ since $x \in \overline{V_n}$ and $f(n)$ is not.



$\square$

Cor Closed intervals $[a,b] \subseteq \mathbb{R}$ are uncountable $\forall a < b$.

TPS Every cts fn $f: X \to Y$ b/w metric spaces, if X is cpt then f is unif cts.

i.e., $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall x_0, x_1 \in X$,

$$d(x_0, x_1) < \delta \implies d(f(x_0), f(x_1)) < \varepsilon.$$

Note δ independent of x_0, x_1!