Day 16

Learning goals

- Introduce compactness
- Bolzano-Weierstrass theorem
- Compactness and Hausdorffness

An analogy:

connectedness : IVT :: compactness : EVT

extreme value theorem

To fully develop this, we'll need the Heine-Borel theorem (compact subsets of $\mathbb{R}$ are closed and bounded) which we'll do on Day 17.

Defn A collection $\mathcal{U}$ of open subsets of X is called an **open cover** of X when $\bigcup_{U \in \mathcal{U}} U = X$. The space X is **compact** when every open cover of X has a finite subcover.

E.g. $\mathbb{R}$ is not compact because $\{(n, n+1) \mid n \in \mathbb{Z}\} \cup$

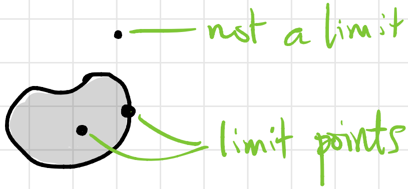
$\{(n+\frac{1}{2}, n+\frac{3}{2}) \mid n \in \mathbb{Z}\}$ has no finite subcover.

Thm If X is compact and $f: X \longrightarrow Y$ cts, then fX is compact. (Here $fX \subseteq Y$ with subspace topology.)

Pf Let $\mathcal{U}$ be an open cover of fX. Then $\mathcal{V} = \{f^{-1}U \mid U \in \mathcal{U}\}$ is an open cover of X. Since X is compact, $\exists$ finite subcover $\mathcal{V}' \subseteq \mathcal{V}$.
Define $\mathcal{U}' = \{U \in \mathcal{U} \mid f^{-1}U \in \mathcal{V}'\}$. Then $\mathcal{U}' \subseteq \mathcal{U}$ is finite and, since every $x \in X$ is in some $f^{-1}U \in \mathcal{V}'$, every $f(x) \in fX$ is in some $U \in \mathcal{U}'$. $\square$

Cor Compactness is a topological property (i.e. it is preserved by homeomorphisms).

Defn A point x is a limit point of X if every nbhd of x contains a point of $X \setminus \{x\}$.



Thm [Bolzano-Weierstrass] Every infinite set in a compact space has a limit point.

{ In a compact set, if you take infinitely many points, those points cluster/accumulate around one of themselves.

Pf Suppose $F \subseteq X$ infinite with no limit points.

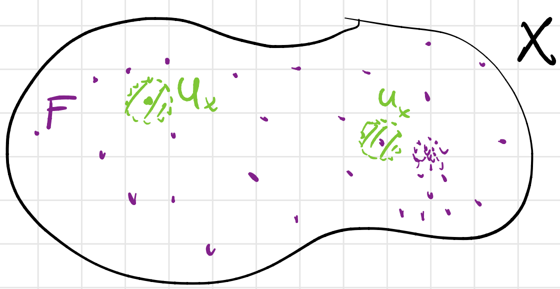
For $x \in X \setminus F$, since x is not a limit point of F, $\exists U_x \subseteq X$ open with $x \in U_x$ and $U_x \cap F = \emptyset$.

For $x \in F$, $\exists U_x \subseteq X$ open with $x \in U_x$ and $U_x \cap F = \{x\}$.

Then $\mathcal{U} = \{U_x \mid x \in X\}$ is an open cover of X.

Since X compact, $\exists x_1, \dots, x_n \in X$ with $U_{x_1} \cup \cdots \cup U_{x_n} = X$. Thus $(U_{x_1} \cup \cdots \cup U_{x_n}) \cap F$

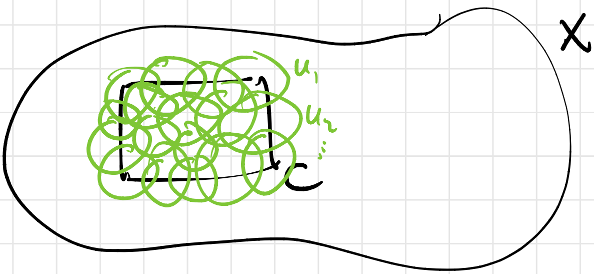
$= F \subseteq \{x_1, \dots, x_n\}$ ⨳. $\square$



Thm Closed subsets of compact spaces are compact.

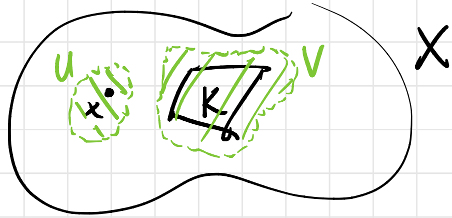
Pf Take X compact, $C \subseteq X$ closed, $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ open cover of C. (Meaning $U_\alpha \subseteq X$ open, $C \subseteq \bigcup_{\alpha \in A} U_\alpha$. Note $\{C \cap U_\alpha \mid \alpha \in A\}$ is an open cover of C in subspace topology.) Then $\mathcal{U} \cup \{X \setminus C\}$ is an open cover of X, thus has a finite subcover $\{U_i \mid 1 \le i \le n\}$ possibly together with $X \setminus C$.

$\llcorner$ finite subcover of C. $\square$

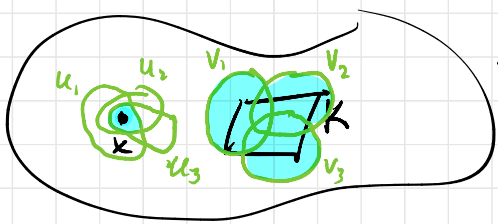


Now consider how compactness and Hausdorffness interact.

Thm Let X be Hausdorff. $\forall x \in X$ and compact $K \subseteq X \setminus \{x\}$ $\exists$ disjoint open $U, V \subseteq X$ with $x \in U$, $K \subseteq V$.



Pf Given such x, K, $\forall y \in K$ take disjoint open $U_y, V_y \subseteq X$ with $x \in U_y$, $y \in V_y$. Then $\{V_y \mid y \in K\}$ is an open cover of K so has a finite subcover $\{V_1, \dots, V_n\}$. Let $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$ — these work! $\square$



Cor Compact subsets of Hausdorff spaces are closed.

Pf Express $X \setminus K$ as the union of the sets U produced above (one for each $x \in X \setminus K$). $\square$

Cor If X is compact and Y is Hausdorff, then every map $f: X \longrightarrow Y$ is closed (i.e. $C \subseteq X$ closed $\Rightarrow fC$ closed).

Pf C is compact so fC is compact so fC is closed. $\square$

Note In the above setup,

- f inj $\Rightarrow$ f embedding
- f surj $\Rightarrow$ f quotient map
- f bij $\Rightarrow$ f homeo.