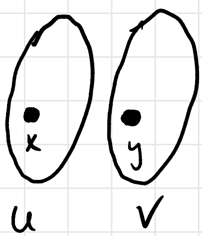
# Day 15

## Learning goals

- Define Hausdorff spaces
- (Non) examples
- Diagonal condition for Hausdorffness
- Other separation conditions (overview)

**Defn** A space $X$ is **Hausdorff** when $\forall x, y \in X$ $\exists$ disjoint open sets $U, V \subseteq X$ with $x \in U$, $y \in V$.





The Hausdorff condition is named after Felix Hausdorff (1868–1942).

Preface to 1914 *Principles of Set Theory*

"of the human privilege of error

to make as economical a use as possible."

Tragically, Hausdorff took his own life in 1942 to avoid being sent to Endenich, a Nazi concentration camp.

**Thm** If $X$ is Hausdorff then every finite subset of $X$ is closed. In particular, points are closed.

**Pf** It suffices to show that every singleton subset $\{x_0\}$ of $X$ is closed. We prove $\overline{\{x_0\}} = \{x_0\}$

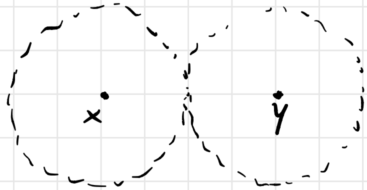
(closure of $\{x_0\}$, $\overline{\{x_0\}} = \bigcap C$, $C \supseteq \{x_0\}$, $C$ closed)

Given $x \in X$, $x \neq x_0$, take $U, V$ disjoint open nbhds of $x, x_0$.

Then $x \notin X \setminus U$, a closed set containing $x_0$, so $x \notin \overline{\{x_0\}}$. $\square$

**E.g.** Every metric space is Hausdorff: Given $x, y \in X$ where $X$ has metric topology wrt $d$, the open balls $B(x, \frac{1}{2}d(x,y))$, $B(y, \frac{1}{2}d(x,y))$

separate $x, y$:




**E.g.** Every order topology is Hausdorff.
(Moral exc)

**E.g.** Any infinite set $X$ with cofinite topology is not Hausdorff:

[$U \subseteq X$ open $\Leftrightarrow$ $X \setminus U$ finite]

$x, y \in X$ separated by disjoint nbhds $U, V \implies$

$$U \cap V = \emptyset \Rightarrow X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V) = X$$

so $X$ is finite. ☹

**E.g.** "Line with two origins" $X = \mathbb{R} \setminus \{0\} \cup \{p, q\}$ is not Hausdorff. Here top on $X$ has basic opens

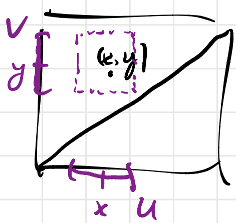
- open intervals not containing 0
- $(-a, 0) \cup \{p\} \cup (0, a)$
- $(-a, 0) \cup \{q\} \cup (0, a)$


**Thm** Product and subspace preserve the Hausdorff condition. Quotient and homotopy equivalence do not preserve the Hausdorff condition.

**Fact** Every top'l space is homeomorphic to the quotient of some Hausdorff space.

**Thm** A space $X$ is Hausdorff iff the diagonal $\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$ is closed.

**Pf** ($\Leftarrow$) Take $x \neq y \in X$. Then $(x,y) \in X \setminus \Delta$ open so $\exists$ basic open $U \times V \subseteq X \setminus \Delta$ containing $(x,y)$. Then $x \in U$, $y \in V$, $U \cap V = \emptyset$, $U, V$ open ✓



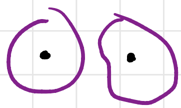

($\Rightarrow$) WTS $X \times X \setminus \Delta$ open. For $(x,y) \in X \times X \setminus \Delta$ know $x, y \in X$, $x \neq y$. Since $X$ is Hausdorff, $\exists U, V \subseteq X$ open disjoint nbhds of $x, y$. Then $U \times V \subseteq X \times X \setminus \Delta$ open in $X \times X$ containing $(x,y)$. $\square$

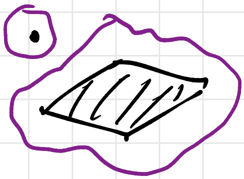
## Other separation properties (/axioms)

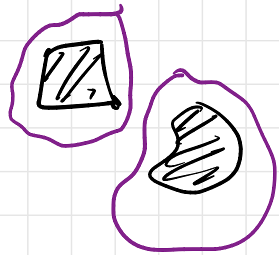
A space is $T_1$ when its points are closed.

A space is regular when it is $T_1$ and $\forall x \in X$, $x \notin B \subseteq X$ closed, $\exists$ disjoint open sets separating $x$ and $B$.

A space is normal when it is $T_1$ and $\forall A, B \subseteq X$ closed disjoint, $\exists$ disjoint open sets containing $A, B$ resp.







$T_1 \supseteq$ Hausdorff $\supseteq$ regular $\supseteq$ normal

aka Fréchet, $T_2$, $T_3$, $T_4$

$\exists$ $T_0$, $T_{2\frac{1}{2}}$, completely $T_2$, $T_{3\frac{1}{2}}$, $T_5$, $T_6$ as well but we won't belabor these.

Urysohn

Tychonoff