Day 14

Learning goals

- Review (path) connectivity
- Topological intermediate value theorem
- Have fun

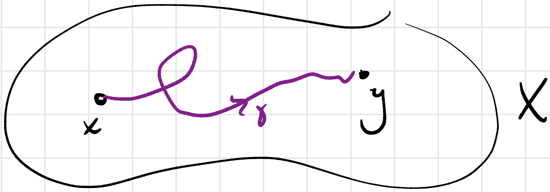
Recall that X is **connected** when all cts maps $f: X \to \{0,1\}$ are constant. Equivalently, one of $U = f^{-1}0$, $V = f^{-1}1$ is empty. Note that f nonconstant $\Rightarrow U, V \neq \emptyset$, $U \cap V = \emptyset$, $U \cup V = X$ i.e. U, V form a **separation** of X.

For X conn'd,

$$\mathrm{Top}(X, \{0,1\}) = \mathrm{Top}(X, \{0\}) \sqcup \mathrm{Top}(X, \{1\}).$$

In fact, X is connected iff $\mathrm{Top}(X, -)$ **preserves coproducts**: $\mathrm{Top}(X, \coprod_{\alpha \in A} Y_\alpha) = \coprod_{\alpha \in A} \mathrm{Top}(X, Y_\alpha)$.

X is **path connected** when $\forall x, y \in X\ \exists \gamma : x \rightsquigarrow y$ in X, i.e. $\gamma : [0,1] \to X$ cts with $\gamma(0) = x$, $\gamma(1) = y$.



We proved that intervals (and rays) in $\mathbb{R}$ are connected, from which it follows that path conn'd $\Rightarrow$ conn'd. (o/w could seperate $[0,1]$)

We proved (path) conn'dness is preserved by

- quotient
- product
- image (of cts fn)
- homotopy equivalence of spaces.

"hard cartoon"

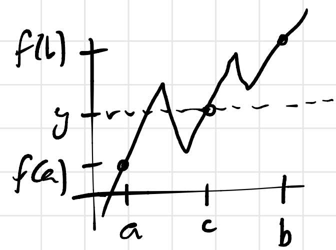
<u>NB</u> The box topology on $\prod_{\alpha \in A} X_\alpha$ has basis $\{ \prod_{\alpha \in A} U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ open } \forall \alpha \in A \}$. This is finer than the product topology whose basic opens have the add'l condition $U_\alpha = X_\alpha$ for almost all α. In HW, you'll see that $\mathbb{R}^{\mathbb{N}}_{box}$ is <u>not</u> conn'd. Of course,

$\mathbb{R}^{\mathbb{N}}_{\text{prod}}$ is conn'd since $\mathbb{R}$ is conn'd and $\prod$ preserves conn'dness.

Intermediate Value Thm

Recall Math 112 IVT:

$f : [a,b] \to \mathbb{R}$ cts, $f(a) \le y \le f(b)$ then $\exists\, c \in [a,b]$ s.t. $f(c) = y$.



Pf $f[a,b]$ is a conn'd subset of $\mathbb{R} \Rightarrow f[a,b]$ is an interval.

Since $f(a), f(b) \in f[a,b]$, get $y \in f[a,b]$. $\square$

We can generalize by considering cts fns $f : X \to Y$ where X is conn'd and Y has the order topology wrt some total order $<$ on Y.

(Basis = open intervals.)

Thm For such $f : X \to Y$, if $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$, then $\exists\, c \in X$ s.t. $f(c) = y$.

Pf The sets $U = fX \cap (-\infty, y)$, $V = fX \cap (y, \infty)$ are nonempty disjoint open subsets of $fX \subseteq Y$.

If $\nexists c \in X$ s.t. $f(c) = y$, then $fX = U \cup V$ is a separation of fX, but fX is conn'd b/c f is cts $\Rightarrow\Leftarrow$

$\square$

E.g. The ordered square is connected but not path conn'd:

$X = [0,1]^2_{lex}$ is the square with lexicographic order topology. Check that we only used the following two properties to show intervals in $\mathbb{R}$ were conn'd:

- least upper bound property
- if $x < y$ $\exists z$ s.t. $x < z < y$.

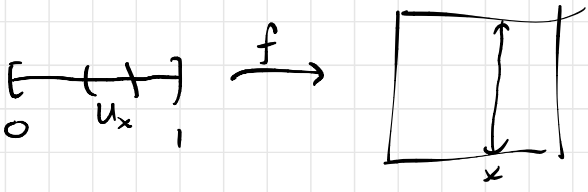
$\Big\}$ linear continuum

X satisfies these as well so is conn'd!

Now suppose for $\Rightarrow\Leftarrow$ $\exists \gamma : (0,0) \rightsquigarrow (1,1)$ in X.

By IVT, $\gamma[0,1] = X$. For each $x \in [0,1]$,

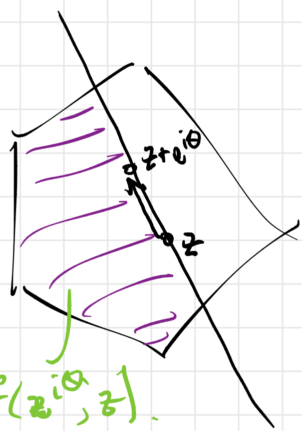
$U_x := f^{-1}(\{x\} \times (0,1))$ is a nonempty open subset of $[0,1]$:



For each $x \in [0,1)$, choose $q_x \in U_x \cap \mathbb{Q}$.
Since the U_x are disjoint, we get an injection $[0,1) \hookrightarrow \mathbb{Q}$, ⨳ since $[0,1)$ uncountable, $\mathbb{Q}$ countable. $\square$

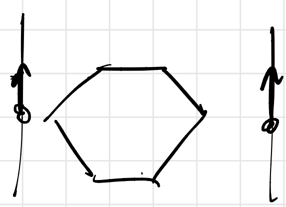
Prop Every convex polygon can be partitioned into two convex polygons, each having equal area and perimeter.

Pf Fix convex polygon $P \subseteq \mathbb{C}$.
Consider the function $f: S^1 \times \mathbb{C} \to \mathbb{R}$ taking $(e^{i\theta}, z)$ to the area to the left of the oriented line through z in direction $e^{i\theta}$:



area $f(e^{i\theta}, z)$.

For any $e^{i\theta}$,
Can have $f(e^{i\theta}, z) = 0$
and area(P):



By IVT, for each $e^{i\theta}$ $\exists z_\theta \in \mathbb{C}$ s.t. $f(e^{i\theta}, z_\theta) = \frac{1}{2}\text{area}(P)$. Call L_θ, R_θ the left and right polygons w/ equal area.

Now consider
$$g : [0, 2\pi] \longrightarrow \mathbb{R}$$
$$\theta \longmapsto \text{perim}(L_\theta) - \text{perim}(R_\theta).$$

WLOG, suppose $g(0) > 0$. Then $g(2\pi) < 0$ (b/c L_θ, R_θ swap). By IVT, $\exists \theta$ s.t. $g(\theta) = 0$.

$\square$