Day 13

Learning Goals

(1) · Connected, path conn'd are htpy invts

(2) · Connectedness vs intermediate value thm

(3) · π_0

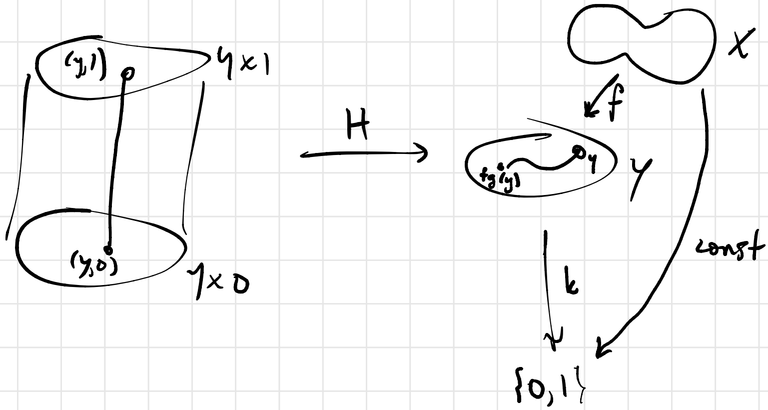
(4) · (Path) connectedness vs product, quotient

(5) · Local conn'dness $\Rightarrow$ path components = conn'd components

(1) **Thm** "Connected" and "path connected" are htpy invts.

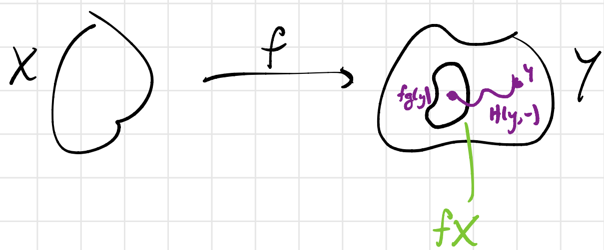
Pf Suppose $f: X \to Y$ is a htpy equiv witnessed by $g: Y \to X$, $H: fg \Rightarrow id_Y$.

Conn'd Suppose X is conn'd, take cts $k: Y \to \{0,1\}$, $y, y' \in Y$. Know $kf: X \to \{0,1\}$ constant, so $kfg(y) = kfg(y')$. The fn $H(y, -): [0,1] \to Y$ is a path from $H(y,0) = fg(y)$ to $H(y,1) = y$, and $kfg(y) = k(y)$ (b/c image of $H(y,-)$ is constant):



Similarly (using $H(y', -)$), $kfg(y') = k(y')$. Since kf is constant on all of X, this must be the same value as $kfg(y) = k(y)$, i.e. $k(y) = k(y')$ so k is constant.

Path conn'd Suppose X is path conn'd. Then fX is path conn'd.



Given $y \in Y \setminus fX$, $H(y,-)$ is a path from $fg(y) \in fX$ to y, so Y is path conn'd. $\square$

(2) We can use connectedness of intervals to prove Intermediate Value Theorem types of results.

Thm (n=1 case of Brouwer fixed point theorem)

Every cts function $f: [-1,1] \to [-1,1]$ has a fixed point.

Pf Suppose for ⨳ $\exists f: [-1,1] \to [-1,1]$ s.t. $\forall x \in [-1,1]$, $f(x) \neq x$. Note $f(-1) > -1$, $f(1) < 1$.

Define
$$g: [-1,1] \to \{\pm 1\}$$
$$x \longmapsto \frac{x - f(x)}{|x - f(x)|}.$$

Then g is cts, $g(-1) = -1$, $g(1) = 1$. But $[-1,1]$ conn'd, so ⨳. $\square$

using that arithmetic operations and composites of cts fns are cts

(3) **Defn** The path components functor

$$\begin{array}{rcl} \pi_0 : \mathrm{Top} & \longrightarrow & \mathrm{Set} \\ X & & \pi_0 X := \text{set of path components of } X \\ \downarrow & \longmapsto & \downarrow \pi_0 f \\ Y & & \pi_0 Y \end{array}$$

where $(\pi_0 f)([x]) = [f(x)]$.

$[x]$ = path comp't of x.

(This is well-defined b/c when $\gamma : x \rightsquigarrow y$ in X, $f\gamma : f(x) \rightsquigarrow f(y)$ in Y.)

Reading The previous proof can be framed as a consequence of functoriality of π_0!

(4) Note that $\amalg$ does not preserve (path) connectedness. We've already seen that quotients do. Subspaces do not.

Thm Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of (path) conn'd spaces. Then $X := \prod_{\alpha \in A} X_\alpha$ is (path) conn'd.

Pf (path conn'd case) Take $a, b \in X$. Choose paths $\gamma_\alpha : a_\alpha \rightsquigarrow b_\alpha$ in each X_α. The function $\gamma : [0,1] \to X$ with components γ_α is a path $a \rightsquigarrow b$.

$$\begin{tikzcd} & {[0,1]} \arrow[dl, "\gamma_\alpha"'] \arrow[d, "\gamma"] \\ X_\alpha & X \arrow[l] \end{tikzcd}$$

$\square$

② As a set, X is the disjoint union of its conn'd components. $X = \coprod_{Z \in \text{conn'd cpts } X} Z$.

There is a natural topology on $\coprod_{\alpha \in A} X_\alpha$ with $U \subseteq \coprod X_\alpha$ open iff $U \cap X_\alpha$ open $\forall \alpha$. (See 1.5.)

It is not the case that $X \cong \coprod_{Z \in \text{conn'd cpts } X} Z$

where RHS has coproduct top.

E.g. $\pi_0 \mathbb{Q} = \{\{r\} \mid r \in \mathbb{Q}\}$ but $\mathbb{Q}$ is not discrete

Nevertheless:

Thm TFAE

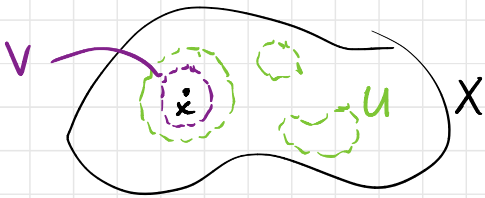
(i) $X \cong \coprod_{Z \in \text{conn'd cpts } X} Z$

(ii) conn'd components of X are open

(iii) $X/\sim$ is discrete for $x \sim y$ when x, y in same conn'd component. $\square$

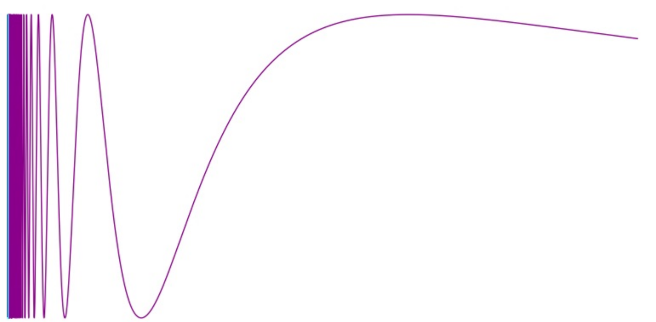
(5) Defn A space X is locally (path) connected when $\forall x \in X$ $\forall$ open nbhd $U \subseteq X$ of x $\exists$ (path) connected open nbhd V of x with $V \subseteq U$.

E.g. Most "nice"/familiar spaces have this property.



The topologist's sine curve does not:

$S = \left(\text{graph of } \sin(1/x) \text{ for } x > 0\right) \cup \left(\{0\} \times [-1,1]\right)$



Have S = closure of image of $(0,\infty)$ so S is conn'd.

Not locally conn'd and not (locally) path connected.

For not path conn'd: $\exists\, t_n \to 0$ with $\sin(t_n) = (-1)^n$ not converging so no path connecting $(0,0)$ to RH pts.

Thm If X is locally path connected, then its connected components and path components are the same.

Pf Read Munkres, §25. (Optional.) $\square$