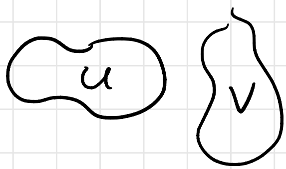
Day 12

Learning Goals

- Connectedness
- Path connectedness
- Path components

Defn A space X is *disconnected* when $\exists\, U, V \subseteq X$ open s.t. (1) $X = U \cup V$, (2) $U \cap V = \varnothing$, and (3) $U, V \neq \varnothing$.



A space X is *connected* when it is not disconnected.

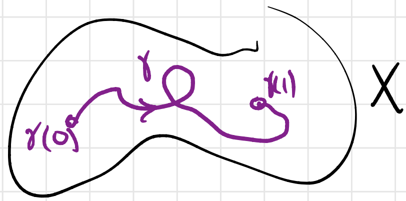
Prop A space X is connected iff every cts fn $X \longrightarrow \{0,1\}_{\text{discrete}}$ is constant.

Pf ($\Leftarrow$) We prove the contrapositive. Suppose X is disconnected with $X = U \cup V$, U, V open and disjoint. Then χ_U is cts and nonconstant.

($\Rightarrow$) Again by contrapositive. Suppose $f : X \to \{0,1\}$ is nonconstant and cts. Then $U = f^{-1}0$, $V = f^{-1}1$

witness that X is disconnected. $\square$

Recall that a path in X is a cts fn $\gamma : [0,1] \to X$



Define an equiv rel'n on X as follows:

$x \sim y$ when $\exists$ path γ in X with $\gamma(0) = x, \gamma(1) = y$.

Reflexive: Constant path at x

Symmetric: Given $\gamma : x \rightsquigarrow y$ define $\bar{\gamma} : y \rightsquigarrow x$ by

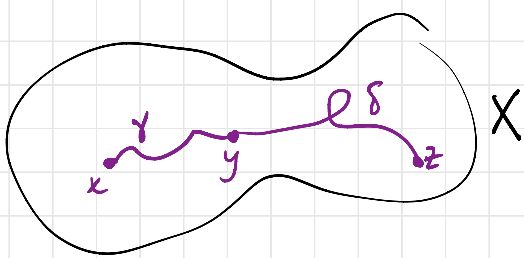
$\bar{\gamma}(t) = \gamma(1-t)$.

informal notation for γ being a path in X from x to y.

Transitive: Given $x \overset{\gamma}{\rightsquigarrow} y \overset{\delta}{\rightsquigarrow} z$ define

$\delta \cdot \gamma : x \rightsquigarrow z$ by

$$(\delta \cdot \gamma)(t) = \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \delta(2t-1) & t \in [1/2, 1] \end{cases}$$



Defn A space X is *path connected* when $\forall x, y \in X$, $x \sim y$ (i.e. $\exists \gamma : x \rightsquigarrow y$).

Thm Image under a cts fn preserves connectedness and path connectedness.

Pf (Connectedness) Suppose $f : X \to Y$ cts and fX disconnected, $g : fX \to \{0,1\}$ nonconstant and cts. Then $gf : X \to \{0,1\}$ is too so X is disconnected.

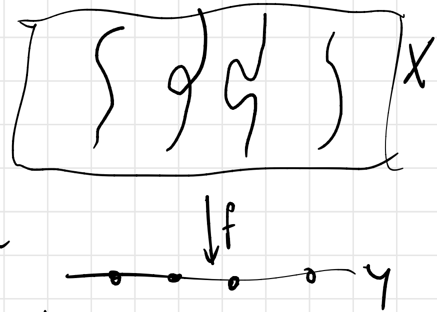
(Path connectedness) Suppose X is path conn'd, $f : X \to Y$ cts. Given $f(x), f(y) \in fX$, choose $\gamma : x \rightsquigarrow y$ in X. Then $f\gamma : f(x) \rightsquigarrow f(y)$. $\square$

Cor Connected and path connected are topological properties. $\square$

Cor The quotient of a (path) connected space is (path) connected.

Pf Quotients are cts surjections. $\square$

Thm Space X, surj fn $f : X \to Y$. If Y is connected in T_f (quotient top) and $f^{-1}y$ connected $\forall y \in Y$, then X is connected.

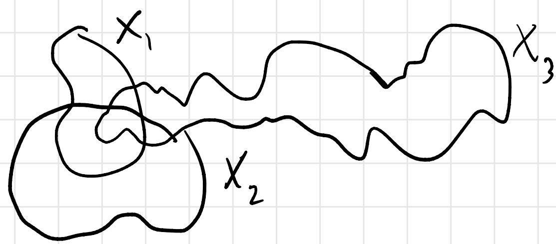


Pf Given cts $g: X \to \{0,1\}$, know g is constant on each $f^{-1}y$. The points of Y may be identified with these fibers, so we get $\bar{g}: Y \to \{0,1\}$ s.t.

$$\begin{tikzcd} & X \arrow[d, "f"] \arrow[dl, "g"'] \\ \{0,1\} & Y \arrow[l, "\bar{g}"] \end{tikzcd}$$

By univ prop of quotient, $\bar{g}$ is continuous. Since Y conn'd, $\bar{g}$ is constant $\Rightarrow$ g constant. □

Thm Suppose $X = \bigcup_{\alpha \in A} X_\alpha$, $\forall \alpha \in A$, X_α is (path) connected, and $\bigcap_{\alpha \in A} X_\alpha \neq \emptyset$. Then X is (path) connected.



E.g. The space $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) is disconnected. Indeed, consider $f := \chi_{\{x \in \mathbb{Q} \mid x > \sqrt{2}\}}$. Then

$f^{-1}\{0\} = \{x < \sqrt{2}\} = (-\infty, \sqrt{2}) \cap \mathbb{Q}$, $f^{-1}\{1\} = (\sqrt{2}, \infty) \cap \mathbb{Q}$.

Thm The connected subsets of $\mathbb{R}$ are the intervals.

Pf Reading: conn'd $\Rightarrow$ interval. $\quad$ $x \le s \le y,\ x, y \in I \Rightarrow s \in I$.

Suppose for contradiction! $I \subseteq \mathbb{R}$ is an interval, $I = U \cup V$, U, V nonempty disjoint. WLOG have $x \in U$, $y \in V$ with $x < y$.

The set $U' = [x, y) \cap U$ is nonempty and bdd above, so $s = \sup U'$ exists by completeness of $\mathbb{R}$.

Since $x < s \le y$ and I is an interval, either $s \in U$ or $s \in V$. If $s \in U$, $\exists \delta > 0$ s.t. $(s-\delta, s+\delta) \subseteq U$ (since U open). If $s \in V$, $\exists \delta > 0$ s.t. $(s-\delta, s+\delta) \subseteq V$ (since V open). In the first case, s is not an upper bound of U', $\lightning$. In the latter, $s-\delta$ is a smaller upper bound of U', $\lightning$. $\square$

Thm Path connected $\Longrightarrow$ connected.

Pf Suppose X is path connected and consider a cts fn $f : X \longrightarrow \{0,1\}$. Fix $x_0 \in X$. $\forall x \in X\ \exists \gamma : x \rightsquigarrow x_0$

Have

$$\begin{tikzcd} & {[0,1]} \arrow[dl, "f\gamma"'] \arrow[d, "\gamma"] \\ \{0,1\} & X \arrow[l, "f"] \end{tikzcd}$$

and $f\gamma$ is constant since $[0,1]$ conn'd. Thus $f(x) = f\gamma(0) = f\gamma(1) = f(x_0)$

$\Longrightarrow f$ is constant. $\square$