Day 11

Learning goals

- Review homotopies
- Introduce the homotopy category
- Understand homotopy equivalence and the notion of homotopy invariance

Given cts fns $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, a **homotopy** $H : f \Rightarrow g$ is a cts fn $H : X \times I \to Y$ s.t. ($I := [0,1]$)

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}\times 1"'] \arrow[dr, bend left, "g"] & \\
X \times I \arrow[r, "H"'] & Y \\
X \arrow[u, "\mathrm{id}\times 0"] \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

{ H is a "movie of functions" interpolating from f to g. }

commutes. Write $f \simeq g$ when a homotopy $H : f \Rightarrow g$ exists.

TPS Is $\simeq$ an equivalence relation on $\mathrm{Top}(X,Y)$?

Note For "nice" X, Y, $\mathrm{Top}(X,Y)$ has a natural topology and the "exponential adjunction" exhibits that a homotopy is equivalent to a path in $Top(X, Y)$.

Write $[f] = \{ g \in Top(X,Y) \mid \exists H: f \Rightarrow g \}$ for the homotopy class of f, and write $[X, Y]$ for $Top(X,Y)/\simeq$, the set of homotopy classes of cts maps $X \longrightarrow Y$.

The homotopy category is $hTop$ with

objects: top'l spaces

morphisms: htpy classes of maps.

I.e., $hTop(X, Y) := [X, Y]$.

If $[g][f] = [gf]$ is well-defined, then $[id_X]$ will serve as the identity.

Moral exercise For $H: f \Rightarrow f'$, $H': g \Rightarrow g'$ check that $H''(x,t) = H'(H(x,t), t)$ defines a homotopy $H'': gf \Rightarrow g'f'$. Thus comp'n of htpy classes is well-defined.

Associativity follows from

$$
\begin{aligned}
([h][g])[f] &= [hg][f] \\
&= [(hg)f] \\
&= [h(gf)] \\
&= [h][gf] \\
&= [h]([g][f]).
\end{aligned}
$$

So hTop really is a category!

A class $[f]$ is an iso in hTop iff $\exists\, [g]$ s.t.

$$[g][f] = [id_X] \quad \& \quad [f][g] = [id_Y].$$

I.e. $\quad gf \simeq id_X \quad \& \quad fg \simeq id_Y$.

E.g. $S^1 \simeq D^2 \setminus \{0\}$ via
$$\begin{array}{ccc} x & \longmapsto & x \\ f: S^1 & \rightleftarrows & D^2 \setminus \{0\} : g \\ \frac{x}{\|x\|} & \longleftarrow\!\shortmid & x \end{array}$$

Note $gf = id_{S^1}$, so may use $H(x,t) = x$ as htpy.

Have $fg(x) = \frac{x}{\|x\|}$. Define
$$H : (D^2 \setminus \{0\}) \times [0,1] \longrightarrow D^2 \setminus \{0\}$$
$$(x,t) \longmapsto tx + (1-t)\frac{x}{\|x\|}$$

parametrizes a straight line from $\frac{x}{\|x\|}$ to x.

Then $H(x,0) = \frac{x}{\|x\|} = fg(x)$ and $H(x,1) = x$

so $H : fg \simeq id_{D^2 \setminus 0}$ as desired.

Have a functor
$$\begin{array}{ccc} \mathrm{Top} & \longrightarrow & \mathrm{hTop} \\ X & & X \\ f\downarrow & \longmapsto & \downarrow [f] \\ Y & & Y \end{array}$$

Functors from hTop are homotopy invariants.

A functor from Top that factors through hTop is a homotopy functor.