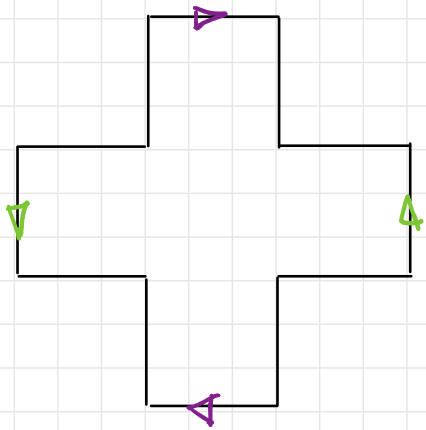
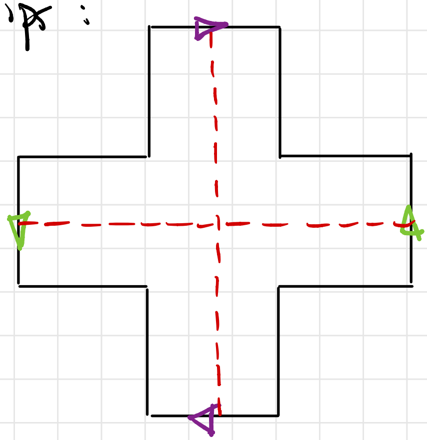
♡ Valentines Day Special

Consider the following identification space:

$V :=$ 

You can think of V as two Möbius strips glued together orthogonally.

Now modify V by deleting the midlines from both strips:

$V' :=$ 

You can make a paper model of V' by taping two Möbius strips together* and then cutting along the midlines.

What do you get?

* To get the most striking result, make sure the "chirality" of the Möbius twists are opposite.

Day 10

Learning goals

- understand "arbitrary" products of sets.
- defn + 1st/2nd char'ns of product topology

Recall For sets X, Y, the Cartesian product of X, Y is $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$.

This has two projection maps

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

satisfying a universal property:

$\forall f_1 : Z \to X,\ f_2 : Z \to Y\ \exists!\ h : Z \to X \times Y$ s.t.

$$\begin{tikzcd} & Z \arrow[ddl, bend right, "f_1"'] \arrow[d, dashed, "\exists! h"] \arrow[ddr, bend left, "f_2"] & \\ & X \times Y \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ X & & Y \end{tikzcd}$$

commutes.

Moral $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ satisfies and is specified by this property.

We can also form the disjoint union of sets:

$$X \sqcup Y := X \times \{1\} \cup Y \times \{2\}$$

$$\begin{tikzcd}
 & X \sqcup Y & \\
X \arrow[ur, "i_1"] & & Y \arrow[ul, "i_2"']
\end{tikzcd}$$

This satisfies the universal property indicated by

$$\begin{tikzcd}
 & Z & \\
 & X \sqcup Y \arrow[u, dashed, "\exists! h"] & \\
X \arrow[ur, "i_1"] \arrow[uur, bend left, "f_1"] & & Y \arrow[ul, "i_2"'] \arrow[uul, bend right, "f_2"']
\end{tikzcd}$$

May form the disjoint union of a collection of sets $\{X_\alpha \mid \alpha \in A\}$ as well:

$$\coprod_{\alpha \in A} X_\alpha := \{(x, \alpha) \mid x \in X_\alpha, \alpha \in A\}$$

$$X_\alpha \xrightarrow{i_\alpha} \coprod_{\alpha \in A} X_\alpha$$

Universal property: $\forall \{f_\alpha : X_\alpha \to Z \mid \alpha \in A\}$ $\exists! h : \coprod X_\alpha \to Z$ s.t. $h i_\alpha = f_\alpha$ $\forall \alpha \in A$.

$$\begin{tikzcd}
 & & Z & \\
 & & \coprod X_\alpha \arrow[u, dashed, "\exists!"] & \\
X_\alpha \arrow[urr] \arrow[uurr, bend left] & X_{\alpha'} \arrow[ur] \arrow[uur] & \cdots & X_{\alpha''} \arrow[ul] \arrow[uul, bend right]
\end{tikzcd}$$

We can now form the product $\prod_{\alpha\in A} X_\alpha$ of a collection of sets $\{X_\alpha \mid \alpha \in A\}$ as the set

$$\left\{ f : A \to \coprod_{\alpha\in A} X_\alpha \;\middle|\; f(\alpha) \in X_\alpha \ \forall \alpha \in A \right\}$$

really $i_\alpha(X_\alpha)$

Universal property:

$$\begin{tikzcd}
& & Z \arrow[ddll, bend right, "f_\alpha"'] \arrow[ddl, "f_{\alpha'}"'] \arrow[d, dashed, "\exists! h"] \arrow[ddr, bend left, "f_{\alpha''}"] & \\
& & \prod_{\alpha\in A} X_\alpha \arrow[dll, "\pi_\alpha"'] \arrow[dl, "\pi_{\alpha'}"] \arrow[dr, "\pi_{\alpha''}"] & \\
X_\alpha & X_{\alpha'} & \cdots & X_{\alpha''}
\end{tikzcd}$$

Now let's get topological: let $\{X_\alpha \mid \alpha \in A\}$ be a collection of topological spaces and consider

$$X = \prod_{\alpha\in A} X_\alpha .$$

<u>Defn</u> The **product topology** on X is the topology gen'd by the basis $\left\{ \prod_{\alpha\in A} U_\alpha \;\middle|\; U_\alpha \subseteq X_\alpha \text{ open and all but finitely many } U_\alpha = X_\alpha \right\}$.

First characterization The product topology on X is the coarsest topology on X for which all of the proj'n maps $\pi_\alpha : X \longrightarrow X_\alpha$ are continuous.

Pf Homework. $\square$

Second characterization The product topology on X is characterized by the following universal property:

$\forall$ function $f : Z \longrightarrow X$, f is cts iff $\forall \alpha \in A$, $\pi_\alpha f : Z \longrightarrow X_\alpha$ is cts.

Pf Homework. $\square$

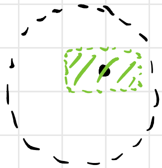
E.g. When is a function $f : Z \longrightarrow \mathbb{R}^2$ cts ($\mathbb{R}^2$ with product top $=$ metric topology)?

Have $f(z) = (f_1(z), f_2(z))$ where

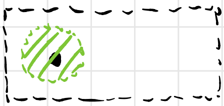
$$\begin{tikzcd}
 & Z \arrow[dl, bend right, "f_1"'] \arrow[d, "f"] \arrow[dr, bend left, "f_2"] & \\
\mathbb{R} & \mathbb{R}^2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathbb{R}
\end{tikzcd}$$

So f is cts iff its **component functions** f_1, f_2 are cts.

Justification of ⊛ :



metric
basis open



product basis
open