Day 9

Learning Goals

- defn and two characterizations of the quotient topology.

Recall X a set, $\sim$ an equiv rel'n on X,

$$X/\sim \; = \{\sim \text{ equiv classes on } X\}$$

The $\sim$ equiv class of $x \in X$ is

$$[x] = [x]_\sim = \{y \in X \mid y \sim x\}.$$

We have a surjection

$$\pi : X \longrightarrow X/\sim$$
$$x \longmapsto [x]$$

Note that $\pi^{-1}[x] = \{y \in X \mid [y] = [x]\} = [x]$.

In fact, every surjective fn $\pi : X \twoheadrightarrow S$ gives rise to an equiv rel'n on X: $x \sim y \iff \pi(x) = \pi(y)$.

We get an iso (of sets)

$$S \longrightarrow X/\sim$$
$$s \longmapsto \pi^{-1}\{s\}.$$

If X is a topological space and $\pi : X \twoheadrightarrow S$, we want to give S a topology for which π is cts.

Defn A set $U \subseteq S$ is open in the quotient topology when $\pi^{-1}U \subseteq X$ is open.

which one...?

First characterization The quotient topology on S is the finest topology on S for which π is cts.

Pf Let $T_{quo} = \{U \subseteq S \mid \pi^{-1}U \text{ open}\}$. This defn directly implies that $\pi : X \to (S, T_{quo})$ is cts. Suppose $\pi : X \to (S, T)$ is also cts. Then $U \in T \Rightarrow \pi^{-1}U$ open $\Rightarrow U \in T_{quo}$, so $T \subseteq T_{quo}$. Thus T_{quo} is the finest such top. $\square$

Second characterization The quotient top on S for $\pi : X \twoheadrightarrow S$ is determined by the following property:

$\forall$ top space Z and $\forall f : S \to Z$, f is cts iff $f\pi$ is cts:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f\pi"] & \\ S \arrow[r, "f"'] & Z \end{tikzcd}$$.

Pf We first check that $\mathcal{T}_{quo}$ satisfies the property.

($\Rightarrow$) Suppose $f : (S, \mathcal{T}_{quo}) \to Z$ is cts. For $U \subseteq Z$ open, we have $(f\pi)^{-1}U = \pi^{-1}(f^{-1}U)$. Since f is cts, $f^{-1}U$ is open, so $\pi^{-1}(f^{-1}U)$ open (by continuity of π).

($\Leftarrow$) Suppose $f\pi$ is cts. Then for $U \subseteq Z$ open, $(f\pi)^{-1}U = \pi^{-1}(f^{-1}U)$ is open in X. By the defn of $\mathcal{T}_{quo}$, this means $f^{-1}U$ open, so f cts.

Now suppose that $\mathcal{T}$ is some other top satisfying the property. <u>WTS</u> $\mathcal{T} = \mathcal{T}_{quo}$.

Consider $f = id_S : (S, \mathcal{T}) \to (S, \mathcal{T}_{quo})$. Then f is cts iff $\pi : X \to (S, \mathcal{T}_{quo})$ is cts; π is cts, so f is cts, whence $\mathcal{T}_{quo} \subseteq \mathcal{T}$.

Similarly, continuity of $id_S : (S, \mathcal{T}) \to (S, \mathcal{T})$ implies $\pi : X \to (S, \mathcal{T})$ cts. The first char'n then gives $\mathcal{T} \subseteq \mathcal{T}_{quo}$, so $\mathcal{T} = \mathcal{T}_{quo}$. $\square$

Note Univ prop tells us that every cts map $S \to Z$ "arises from" a cts map $X \to Z$ which is constant on the fibers of π.

i.e. $\pi^{-1}\{s\}$

Defn Call a cts fn $f: X \twoheadrightarrow Y$ a **quotient map** when the top on Y equals the quotient top on Y rel to f.

E.g.

- $$\pi : [0,1] \longrightarrow S^1, \quad t \longmapsto (\cos(2\pi t), \sin(2\pi t))$$ is a quotient map.

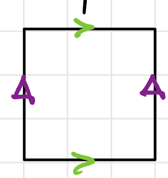
  Thus cts fns $S^1 \to Z$ may be identified with cts $\gamma : [0,1] \longrightarrow Z$ s.t. $\gamma(0) = \gamma(1)$.

- Consider equiv rel'n $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ given by $x \sim \lambda x$ for $\lambda \in \mathbb{R}^\times$. Then **projective space**
  $$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$
  space of lines through 0 in $\mathbb{R}^{n+1}$

  is the space of lines through 0 in $\mathbb{R}^{n+1}$.

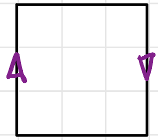
- "Identification spaces" are quotient spaces:

  torus $$T = I^2 \Big/ \begin{matrix} (x,0) \sim (x,1) \\ (0,y) \sim (1,y) \end{matrix}$$

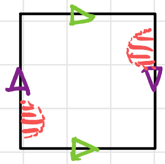


$I = [0,1]$

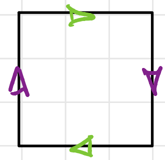
Möbius band



Klein bottle



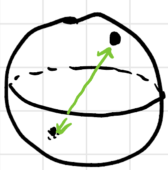
$\mathbb{RP}^2$



Why is this homeomorphic to $\mathbb{R}^3 \setminus \{0\} / x \sim \lambda x$?

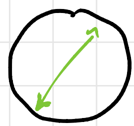
$$S^2 / x \sim -x \cong \mathbb{RP}^2$$

$$\left\{ \frac{x}{|x|}, \frac{-x}{|x|} \right\} \mapsfrom [x]$$

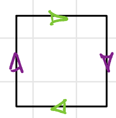


Unique representatives of equiv classes in upper hemisphere except along the equator.

So $\mathbb{RP}^2 \cong D^2 / x \sim -x$ for $x \in \partial D^2$



"Smoosh" into a square and you get



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