Day 8

Learning goals

- How we'll present defns/characterizations of topologies
- Subspace topology

Presently want to build new topologies from old:

subspace: $Y \subseteq X$, Y inherit top from X

quotient: $X/\sim$ (for $\sim$ an equiv rel'n on X)

product: $X \times Y$ and $\prod_{i \in I} X_i$

coproduct: $X \amalg Y$ (disjoint union)

Proceed in three stages:

- classic defn: explicit (and unmotivated)
- first characterization: coarsest or finest topology for which maps into or out of space are cts
- second characterization: universal property

Defn For $(X, \mathcal{T}_X)$ top space and $Y \subseteq X$, define the subspace topology on Y (relative to X)

to be $\mathcal{T}_Y := \{ U \cap Y \mid U \in \mathcal{T}_X \}$.

First char'n: More generally, consider $f : S \to X$, X space, S a <u>set</u>. Is there a <u>coarsest</u> topology on S s.t. f is cts?

Yes: the intersection of all top's on S for which f is cts. (Note: nonempty since f is cts wrt discrete top on S).

Call this int'n $\mathcal{T}_f$ and note $\mathcal{T}_f = \{ f^{-1} U \mid \underset{\text{open}}{U \subseteq X} \}$.

For $Y \subseteq X$ and $i : Y \to X$ the inclusion function,
$y \mapsto y$

we have $\mathcal{T}_Y = \mathcal{T}_i$ since $i^{-1} U = U \cap Y$.

More generally, call $\mathcal{T}_f$ the subspace topology on S whenever $f : Y \to X$ is injective.

Defn A cts injection $f : Y \to X$ is an embedding when the top on Y matches $\mathcal{T}_f$.

Note $\mathrm{id} : (\mathbb{R}, \mathcal{T}_{disc}) \to (\mathbb{R}, \mathcal{T}_{std})$ is a cts injection but <u>not</u> an embedding.

Second char'n:

<u>Thm</u> For $(X, \mathcal{T}_X)$ a top space, $Y \subseteq X$, $i: Y \to X$ the inclusion map, the subspace top on Y is characterized by the following universal property:
$\forall$ space $(Z, \mathcal{T}_Z)$ and function $f: Z \to Y$,
f is cts iff $if: Z \to X$ is cts.

<u>Pf</u> Need to show (1) subspace top satisfies this property
(2) subspace top is characterized by this property.

(1): For $\mathcal{T}_Y$ subspace top on Y and $(Z, \mathcal{T}_Z)$ top space, $f: Z \to Y$ any fn, must show
f cts iff if cts.

($\Rightarrow$) if f cts, then if is a comp'n of cts fns hence cts.

($\Leftarrow$) Suppose $if: Z \to X$ cts, $U \subseteq Y$ open. Then
$U = V \cap Y = i^{-1}V$ for some $V \subseteq X$ open
(by defn of subspace top). Since if cts,
$(if)^{-1} V = f^{-1} U$ is open $\Rightarrow f$ cts.

(2) Now suppose $\mathcal{T}'$ is some top on Y satisfying the univ property. WTS $\mathcal{T}' = \mathcal{T}_Y$.

Univ prop says

$$\begin{tikzcd} Z \arrow[r, dashed, "f"] \arrow[dr, "if"'] & Y \arrow[d, "i"] \\ & X \end{tikzcd}$$

f cts ~~iff~~ if if cts

Take $Z = (Y, \mathcal{T}_Y)$, $Y = (Y, \mathcal{T}')$, $f = id_Y$:

$$\begin{tikzcd} (Y, \mathcal{T}_Y) \arrow[r, dashed, "id_Y"] \arrow[dr, "i = i\, id_Y"'] & (Y, \mathcal{T}') \arrow[d, "i"] \\ & X \end{tikzcd}$$

Since $i\, id_Y = i$ is cts, learn that $id_Y : (Y, \mathcal{T}_Y) \to (Y, \mathcal{T}')$ is cts $\Longrightarrow$ $\mathcal{T}' \subseteq \mathcal{T}_Y$.

(make sure you understand this!

To prove $\mathcal{T}_Y \subseteq \mathcal{T}'$, consider

$$\begin{tikzcd} (Y, \mathcal{T}') \arrow[r, "id_Y"] \arrow[dr, "i"'] & (Y, \mathcal{T}') \arrow[d, "i"] \\ & X \end{tikzcd}$$

Since id_Y cts, learn that i is cts. By first char'n,

$\mathcal{T}_Y$ is the coarsest top for which i is cts,

so $\mathcal{T}_Y \subseteq \mathcal{T}'$. $\square$

TPS Is the subspace top on $\mathbb{Q} \subseteq \mathbb{R}$ discrete?

(No: $\{r\}$ open in $\mathbb{Q}^{disc}$, not in $\mathbb{Q}^{sub}$ which has basis $\{\mathbb{Q} \cap (a,b)\}$.)