

Day 8

Learning goals

- How we'll present defns/characterizations of topologies
- Subspace topology

Presently want to build new topologies from old:

subspace: $Y \subseteq X$, Y inherit top from X

quotient: X/\sim (for \sim an equiv rel'n on X)

product: $X \times Y$ and $\prod_{i \in I} X_i$

coproduct: $X \sqcup Y$ (disjoint union)

Proceed in three stages:

- classic defn: explicit (and unmotivated)
- first characterization: coarsest or finest topology for which maps into or out of space are cts
- second characterization: universal property

Defn For (X, \mathcal{T}_X) top space and $Y \subseteq X$, define the subspace topology on Y (relative to X)

to be $\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}_X\}$.

First char'n: More generally, consider $f: S \rightarrow X$, X space, S a set. Is there a coarsest topology on S s.t. f is cts?

Yes: the intersection of all top's on S for which f is cts. (Note: nonempty since f is cts wrt discrete top on S).

Call this int'n \mathcal{T}_f and note $\mathcal{T}_f = \{f^{-1}U \mid U \in X \text{ open}\}$.

For $Y \in X$ and $i: Y \rightarrow X$ the inclusion function, $y \mapsto y$

we have $\mathcal{T}_Y = \mathcal{T}_i$ since $i^{-1}U = U \cap Y$.

More generally, call \mathcal{T}_f the subspace topology on S whenever $f: Y \rightarrow X$ is injective.

Defn A cts injection $f: Y \rightarrow X$ is an **embedding** when the top on Y matches \mathcal{T}_f .

Note $\text{id}: (\mathbb{R}, \mathcal{T}_{\text{disc}}) \rightarrow (\mathbb{R}, \mathcal{T}_{\text{std}})$ is a cts

injection but not an embedding.

Second char'n:

Thm For (X, \mathcal{T}_X) a top space, $Y \subseteq X$, $i: Y \rightarrow X$ the inclusion map, the subspace top on Y is characterized by the following universal property:
 \forall space (Z, \mathcal{T}_Z) and function $f: Z \rightarrow Y$,
 f is cts iff $if: Z \rightarrow X$ is cts.

Pf Need to show (1) subspace top satisfies this property
(2) subspace top is characterized by this property.

(1): For \mathcal{T}_Y subspace top on Y and (Z, \mathcal{T}_Z) top space, $f: Z \rightarrow Y$ any fn, must show
 f cts iff if cts.

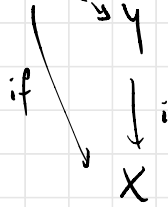
(\Rightarrow) if f cts, then if is a comp'n of cts fns
hence cts.

(\Leftarrow) Suppose $if: Z \rightarrow X$ cts, $U \subseteq Y$ open. Then

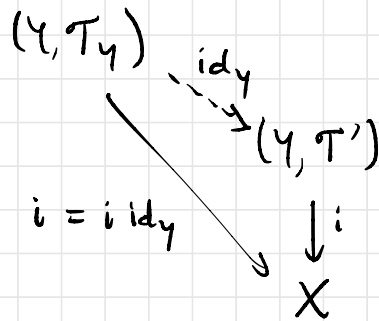
$U = V \cap Y = i^{-1}V$ for some $V \subseteq X$ open
(by defn of subspace top). Since if cts,
 $(if)^{-1}V = f^{-1}U$ is open $\Rightarrow f$ cts.

(2) Now suppose T' is some top on Y satisfying the univ property. WTS $T' = T_Y$.

Univ prop says $Z \xrightarrow{f} Y$ is cts iff if cts



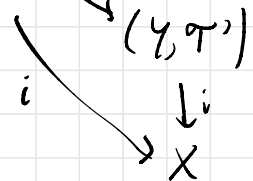
Take $Z = (Y, T_Y)$, $Y = (Y, T')$, $f = \text{id}_Y$:



Since $i \text{id}_Y = i$ is cts, learn that $\text{id}_Y: (Y, T_Y) \rightarrow (Y, T')$ is cts $\Rightarrow T' \in T_Y$.

(make sure you understand this!)

To prove $T_Y \in T'$, consider $(Y, T') \xrightarrow{\text{id}_Y} (Y, T')$



Since id_Y is cts, learn that i is cts. By first char'n,

τ_Y is the coarsest top for which i is cts,
so $\tau_Y \subseteq \tau'$. □

TPS Is the subspace top on $\mathbb{Q} \subseteq \mathbb{R}$ discrete?
(No: $\{r\}$ open in \mathbb{Q}^{disc} , not in \mathbb{Q}^{sub} which
has basis $\{\mathbb{Q} \cap (a,b)\}$.)