Day 7

Learning Goals

- Reorient ourselves in the world of topology
- More examples of spaces
- More examples of continuous functions.

Recall A topological space is a pair $(X, \mathcal{T})$ with X a set, $\mathcal{T} \subseteq 2^X$ s.t.

- $\emptyset, X \in \mathcal{T}$
- $\mathcal{T}$ closed under arbitrary unions
- $\mathcal{T}$ closed under finite intersections

More examples of spaces

(1) May consider the cofinite topology on any set X: $U \subseteq X$ open iff $X \setminus U$ finite (or $U = \emptyset$).

(2) $\emptyset$ and $*$ have unique topologies.

Note $\forall$ space X, $\exists!$ cts fns $\underset{\text{initial}}{\emptyset} \to X \to \underset{\text{terminal}}{*}$.

(3) Some topologies on $\mathbb{R}$:

- metric — we mean this unless o/w stated
- discrete
- cofinite

- **lower limit topology**: basis $\{[a,b) \mid a<b \in \mathbb{R}\}$.

(4) For X totally ordered, the **order topology** on X has basis $\{(a,b) \mid a<b\in X\} \cup \{(a,\infty) \mid a\in X\}$
$\cup \{(-\infty, b) \mid b \in X\}$.

(5) Typically think of $\mathbb{Z}$ as discrete, but here is a novel topology: for $a\in\mathbb{Z}\setminus\{0\}$, $b\in\mathbb{Z}$ let

$$S(a,b) = \{an+b \mid n\in\mathbb{N}\}.$$ arithmetic sequence

E.g. $S(3,-4) = \{-4,-1,2,5,8,11,\dots\}$

Let $\mathcal{B} = \{S(a,b) \mid a\in\mathbb{Z}\setminus\{0\},\ b\in\mathbb{Z}\}$.

Claim $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

Indeed, for $x\in\mathbb{Z}$, $S(1,0)\ni 1x+0=x$, and if $x\in S(a,b)\cap S(c,d)$ then

$$x\in S(\operatorname{lcm}(a,c),x) \in S(a,b)\cap S(c,d)$$

$S(2,1)$ $S(3,0)$ $S(6,3)$

So may take unions of arithmetic sequences (and $\emptyset$) as opens for $\mathbb{Z}$.

TPS Each basic open $S(a,b)$ is also closed.

Now note that

$$\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{p \text{ prime}} S(p,0)$$

b/c every integer $\neq \pm 1$ has a prime factor. Furthermore, $\{\pm 1\}$ is finite so not open, whence $\mathbb{Z} \setminus \{\pm 1\}$ is not closed. In particular, $\mathbb{Z} \setminus \{\pm 1\}$ is not a finite union of closed sets, so there are infinitely many prime numbers!

(This proof is due to Furstenberg, 1955 — published when he was an undergrad.)

(6) R a comm ring, $\operatorname{Spec} R = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ prime ideal}\}$ with closed sets $VE := \{\mathfrak{p} \in \operatorname{Spec} R \mid E \subseteq \mathfrak{p}\}$, E any subset of R.

(7) Read: Norms $\rightsquigarrow$ metrics $\rightsquigarrow$ top.

<u>More examples of cts fns</u>

(1) $S = \{0,1\}$ with topology $\{\emptyset, \{1\}, S\}$ is the Sierpinski space. For $U \subseteq X$ open, $\chi_U : X \longrightarrow S$

$$x \longmapsto \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

is cts. In fact, every cts fn $f: X \to S$ is of this form: $f = \chi_{f^{-1}\{1\}}$. Thus $\mathrm{Top}(X,S) \cong T_X$.

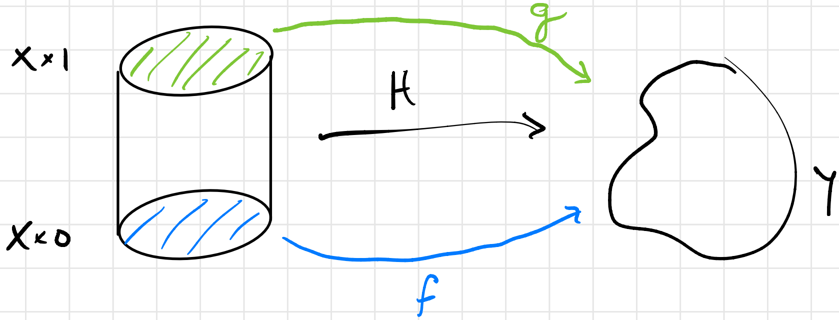
(2) $\mathrm{Top}(*, X) \cong X$

(3) Later: $\mathrm{Top}(\ \ , \{0,1\}^{disc})$ detects connectedness.

(4) A path in X is a cts fn $\gamma : [0,1] \longrightarrow X$.

A loop in X is a cts fn $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = \gamma(1)$.

(5) A homotopy between $f, g : X \longrightarrow Y$ is a cts fn $H : X \times [0,1] \longrightarrow Y$ s.t. $H(-,0) = f$ and $H(-,1) = g$. Write $H : f \simeq g$.



$\therefore$ { H is a "movie" of fns starting with f, ending with g.

Note Need to know $X \times [0,1]$ as a space to make this precise — product topology.

Spaces X, Y are homotopy equivalent when $\exists$ cts fns $f : X \rightleftarrows Y : g$ s.t. $gf \simeq id_X$ and $fg \simeq id_Y$. Write $X \simeq Y$, call f a homotopy equiv.

E.g. $S^1 \simeq D^2 \setminus \{0\}$ via $f : S^1 \rightleftarrows D^2 \setminus \{0\} : g$

$$x \longmapsto x$$
$$\frac{x}{\|x\|} \longleftarrow\!\shortmid x$$

Note $gf = id_{S^1}$, so may use $H(x,t) = x$ as htpy.

Have $fg(x) = \frac{x}{\|x\|}$. Define

$$H : (D^2 \setminus \{0\}) \times [0,1] \longrightarrow D^2 \setminus \{0\}$$
$$(x,t) \longmapsto tx + (1-t)\frac{x}{\|x\|}$$

parametrizes a straight line from $\frac{x}{\|x\|}$ to x.

Then $H(x,0) = \frac{x}{\|x\|} = fg(x)$ and $H(x,1) = x$

so $H : fg \simeq \mathrm{id}_{D^2 \setminus 0}$ as desired.

May define the homotopy category hTop (or Ho(Top) or Hot) to have objects top'l spaces and morphisms homotopy classes of cts fns. The isos in hTop are homotopy equivalences.

What's next?

Make sense of subspace, product, and quotient topologies. We will then be able to define things like

$$\mathrm{UConf}_n(X) = \underbrace{X^n \setminus \Delta}_{\Delta\text{: fat diagonal}} / \underbrace{\mathfrak{S}_n}_{\text{symmetric group acting by permuting factors}}$$

the unordered configuration space of n-subsets of X.

- Paths in $\mathrm{UConf}_n(X)$ will be trajectories of these n non-colliding points.

- Homotopy classes of paths will be "topologically distinct" trajectories.

- Homotopy classes of (based) loops in $UConf_n(X)$ form the n-th braid group of X.

E.g. With $X = \mathbb{R}^2$, get the Artin braid group.