Day 6

Learning goals:

- Functors
  - defn
  - examples
  - adjectives
  - application

In category thy, we want to compare objects. Categories themselves are a type of object. How do we compare them?

Defn A *functor* F from a category C to a category D (written $F: C \longrightarrow D$) is

(i) $\forall X \in \mathrm{ob}\, C$, an object $FX \in \mathrm{ob}\, D$

(ii) $\forall f: X \longrightarrow Y$ in C, a morphism

$$Ff: FX \longrightarrow FY \text{ in } D.$$

s.t. (iii) $F(gf) = (Fg)(Ff)$ $\forall\ X \xrightarrow{f} Y \xrightarrow{g} Z$ in C

and (iv) $F\mathrm{id}_X = \mathrm{id}_{FX}$ $\forall X \in \mathrm{ob}\, C$.

Such a functor is sometimes called a *covariant functor*. A functor $C^{op} \longrightarrow D$ is a *contravariant functor* from C to D. these "reverse arrows".

E.g. (1) For $X \in \operatorname{ob} C$ have the functor $C(X,-)$:

$$\begin{array}{ccc} Y & & C(X,Y) \\ f\downarrow & \longmapsto & \downarrow f_* \\ Z & & C(X,Z) \end{array}$$

from C to Set

(2) $C(-,X)$ is a contravariant functor from C to Set:

$$\begin{array}{ccc} Y & & C(Y,X) \\ f\downarrow & \longmapsto & \uparrow f^* \\ Z & & C(Z,X) \end{array}$$

(3) Suppose that the objects of C consist of a set + more structure and that the morphisms of C are functions on said sets satisfying add'l properties. (E.g. $C = \mathrm{Vect}_k$, Top, Gp, Ring, CRing, ...)

Have a forgetful functor

$$U : C \longrightarrow \mathrm{Set}$$

taking $X \in \operatorname{ob} C$ to its underlying set and a morphism to its underlying function.

(4) In the above setting, typically have a free functor $F : \mathrm{Set} \longrightarrow \mathcal{C}$. In case $\mathcal{C} = \mathrm{Vect}_k$, have $FS = k^S = \{\text{functions } S \to k\}$.

May think of this as a vector space with basis S (more formally, $\{\chi_s \mid s \in S\}$ is a basis for $\chi_s : S \to F$ the characteristic function of s). We have

$$
\begin{array}{ccccc}
\mathrm{Set} & \xrightarrow{F} & \mathrm{Vect}_k & & \\
S & & k^S & \chi_s & \\
g\downarrow & \longmapsto & \downarrow & \downarrow & \\
T & & k^T & \chi_{g(s)} &
\end{array}
$$

extend this assignment linearly

Free and forgetful functors fit into a free-forgetful adjunction that we will study later.

(5) The power set gives a contravariant functor from Set to Set:

$$
\begin{array}{ccccc}
S & & 2^S & f^{-1}U := \{s \in S \mid f(s) \in U\} \\
f\downarrow & \longmapsto & \uparrow f^{-1} & \uparrow \\
T & & 2^T & U
\end{array}
$$

(So maybe we should write 2^f for f^{-1} in order to not confuse it with an inverse function?)

Note that $id_S^{-1} = id_{2^S}$ and for $S \xrightarrow{f} T \xrightarrow{g} U$,

$$(gf)^{-1}V = \{s \in S \mid gf(s) \in V\}$$
$$= \{s \in S \mid f(s) \in g^{-1}V\}$$
$$= f^{-1}[g^{-1}V].$$

This proves that $2^{(\,)}$ is a functor.

__Lemma__ If $F: C \longrightarrow D$ is a functor and $f: X \longrightarrow Y$ is an iso in C, then Ff is an iso in D.

__Pf__ Suppose $g: Y \longrightarrow X$ is an inverse to f in C:

$$gf = id_X, \quad fg = id_Y.$$

Then $F(gf) = F\,id_X$ and $F(fg) = F\,id_Y$

$$\Rightarrow (Fg)(Ff) = id_{FX} \qquad \Rightarrow (Ff)(Fg) = id_{FY}$$

Thus Fg is inverse to Ff, so Ff is an iso. $\square$

To study topological properties, we will want "good" functors $F: \mathrm{Top} \longrightarrow \mathcal{C}$. Note that $FX \not\cong FY \Rightarrow X \not\cong Y$, so we can distinguish top'l spaces w/ such tools.

Note that a functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ induces a set map

$$\mathcal{C}(X,Y) \xrightarrow{\circledast} \mathcal{D}(FX, FY) \quad \forall X, Y \in \mathrm{ob}\,\mathcal{C}.$$
$$f \longmapsto Ff$$

Defn
- Call F faithful if $\circledast$ is injective $\forall X, Y$.
- Call F full if $\circledast$ is surjective $\forall X, Y$.
- Call F fully faithful if $\circledast$ is bijective $\forall X, Y$.

If F is injective on objects and fully faithful, call it a full embedding of categories.

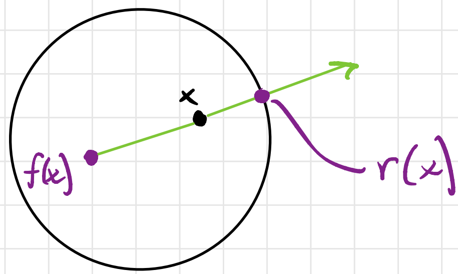
So what can you do with a functor? We will later study the fundamental group functor

$$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Gp}.$$

Facts $\pi_1(D^2) = 0$, $\pi_1(S^1) \cong \mathbb{Z}$

Brouwer Fixed Point Theorem Every cts function $f: D^2 \to D^2$ has a fixed point (i.e. $\exists x \in D^2$ s.t. $f(x) = x$).

Pf Suppose for contradiction that there exists a cts fn $f: D^2 \to D^2$ with no fixed points.
Then the following construction defines a cts fn $r: D^2 \to S^1$:



If $i: S^1 \to D^2$ is the inclusion of the boundary, then $ri = id_{S^1}$ b/c r is the identity on ∂D^2.
Thus

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r, "\pi_1(i)"] \arrow[rr, bend left, "id"] & \pi_1(D^2) \arrow[r, "\pi_1(r)"] & \pi_1(S^1) \\
\mathbb{Z} \arrow[u, phantom, "\cong" rotate=90] \arrow[rr, bend right, "0"] & 0 \arrow[u, phantom, "\cong" rotate=90] & \mathbb{Z} \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

$\square$