Day 5 (ctd)

Learning Goals

- Isomorphisms
  - definition
  - examples (Top, Set, BG, Vect_k)
- Detecting isomorphisms with $C(,)$.

see previous notes

TPS For a group G, what are the isos in BG? (This explains the term "groupoid".)

E.g. • An iso of vector spaces is an iso on underlying sets and hence a bijection. Moreover, the inverse function of any linear bij'n is linear, so isos in Vect_k are the same as linear bij'ns.

• In $\mathrm{FinVect}_k$, choosing a basis of V makes $V \cong k^n$ for $n = \dim V \in \mathbb{N}$. The linear transf'ns $k^m \longrightarrow k^n$ are in bijective correspondence with $n \times m$ matrices. Later, we will see that this makes Mat_k a skeleton of $\mathrm{FinVect}_k$.

Idea To understand $X \in \operatorname{ob} C$, compare it with other objects of C.

Defn For $f: X \to Y$ morphism in C and $Z \in \operatorname{ob} C$, get

$$C(Z,X) \xrightarrow{f_*} C(Z,Y)$$

$$\begin{tikzcd} X \\ Z \arrow[u, "g"] \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[r, "f"] & Y \\ Z \arrow[u, "g"] \arrow[ur, "f_*(g) := fg"'] \end{tikzcd}$$

the pushforward of f (given by postcomposition with f). Similarly

$$C(X,Z) \xleftarrow{f^*} C(Y,Z)$$

$$\begin{tikzcd} & Z \\ X \arrow[r, "f"'] \arrow[ur, "f^*(g) := gf"] & Y \arrow[u, "g"'] \end{tikzcd} \longmapsfrom \begin{tikzcd} Z \\ Y \arrow[u, "g"'] \end{tikzcd}$$

the pullback of f (given by precomposition with f).

The following are equivalent

Thm TFAE:

- $f: X \to Y$ is an isomorphism
- $\forall Z \in \operatorname{ob} C$, $f_*: C(Z,X) \to C(Z,Y)$ is an iso of sets

• $\forall Z \in \operatorname{ob} C$, $f^*: C(Y,Z) \longrightarrow C(X,Z)$ is an iso of sets.

Slogan An object is determined (up to iso) by its relationship with other objects.

Pf Thm (1) $\Leftrightarrow$ (2) : Suppose $f: X \to Y$ an iso with inverse $g: Y \longrightarrow X$. Claim : $g_*: C(Z,Y) \to C(Z,X)$ is inverse to $f_*: C(Z,X) \longrightarrow C(Z,Y)$.

Indeed, $g_* f_* h = g_*(fh) = g(fh) = (gf)h = \mathrm{id}\, h$
$= h$

and $f_* g_* k = f_*(gk) = f(gk) = (fg)k = k$.

Now suppose $f_*: C(Z,X) \to C(Z,Y)$ is an iso $\forall Z$.

For $Z = Y$, get $f_*: C(Y,X) \xrightarrow{\cong} C(Y,Y)$

$$\exists\, g \longmapsto \mathrm{id}_Y = f_* g$$

$\Rightarrow fg = \mathrm{id}_Y$.

For $Z = X$, get $f_*: C(X,X) \xrightarrow{\cong} C(X,Y)$

$$\mathrm{id}_X \longmapsto f_* \mathrm{id}_X = f$$

and $f_*(gf) = (fg)f = f$. Thus injectivity of f_*

implies $gf = id_X$.

Moral exc: Other equivs. $\square$

E.g. Let S be the Sierpinski space with $S = \{0,1\}$ and open sets $\{\emptyset, \{0\}, \{0,1\}\}$.

Then $Top(X, S) \cong T_X$ (the set of open sets for X)

$$\begin{tikzcd} X \arrow[d, "g"'] \\ S \end{tikzcd} \longmapsto g^{-1}\{0\}.$$

Given $f: X \to Y$, the pullback

$$f^*: Top(Y,S) \to Top(X,S)$$

is identified w/
$$\begin{aligned} T_Y &\to T_X \\ U &\mapsto f^{-1}U. \end{aligned}$$

Thus f an iso $\Longrightarrow f^*: T_Y \to T_X$ is a bij'n.

Q Converse? (Only assuming f^*_S an iso, not $f^*_Z \ \forall Z$.)

A No! $\exists$ cts bij'ns which are not homeomorphisms.

E.g. $[0,1) \to S^1$